ON A PROBLEM OF BLOCH AND NEVANLINNA

WALTER RUDIN

In [2, p. 138] the question is raised (and attributed to Bloch) whether there exists a bounded function, analytic in the unit circle, whose derivative is not of bounded characteristic. Frostman [1, p. 181] has answered the question affirmatively by constructing a Blaschke product whose derivative is of unbounded characteristic; this product is of course not continuous on the boundary of the unit circle.

The theorem of the present note furnishes an example of an absolutely convergent power series (with Hadamard gaps) whose derivative is not of bounded characteristic; this follows from the fact that every function of bounded characteristic has finite radial limits along almost all radii. It also gives a simple example of an analytic function which tends to infinity along almost all radii.

THEOREM. *There exists a power series* $f(z)=\sum a_k z^{n_k}$ *such that* $\sum |a_k| < \infty$, $n_k/n_{k-1}\to\infty$, *and* $\lim_{r\to 1} f'(re^{i\theta})=\infty$ *for almost all* θ.

The construction on which the proof is based is similar to one suggested to me by Professor Erdös in connection with a different problem.

Take $n_1=1$; having chosen $n_1, \cdots, n_{k-1}$ $(k\geqq 2)$, choose a positive integer n_k such that

$$n_k > k^2 n_{k-1}^2, \tag{1}$$

$$\sum_{s=1}^{k-1} n_s\left[1-\left(1-n_k^{-1/2}\right)^{n_s}\right] < 1, \tag{2}$$

and define

$$f(z)=\sum_{k=1}^{\infty} k^{-2} z^{n_k}, \qquad g(z)=zf'(z)=\sum_{k=1}^{\infty} k^{-2} n_k z^{n_k}.$$

Put $r_k = 1-n_k^{-1/2}$, and let J_k be the interval $[r_{k+1}, r_{k+2}]$. For any $A<\infty$ (fixed from now on) let T_k be the set of all θ such that $|g(re^{i\theta})| < A$ for some $r\in J_k$, and let E_A be the set on which

$$\liminf_{r\to 1} |g(re^{i\theta})| < A.$$

Presented to the Society, December 29, 1954; received by the editors August 2, 1954.

It is clearly sufficient to prove that $m(E_A)=0$. We put

$$P_k(r,\theta) = \sum_{s=1}^{k} s^{-2} n_s e^{in_s\theta}(1-r^{n_s}),$$

$$Q_k(r,\theta) = \sum_{s=k+2}^{\infty} s^{-2} n_s r^{n_s} e^{in_s\theta},$$

$$R_k(\theta) = \sum_{s=1}^{k} s^{-2} n_s e^{in_s\theta}.$$

By (2) we have

(3) $$\left| P_k(r,\theta) \right| < \sum_{s=1}^{k} n_s(1-r_{k+1}^{n_s}) < 1 \qquad (r \in J_k).$$

Keeping in mind that $n(1-n^{-1/2})^n \to 0$ as $n \to \infty$, we see that

(4) $$\left| Q_k(r,\theta) \right| \leqq \sum_{k+2}^{\infty} s^{-2} n_s r_{k+2}^{n_s} < \sum_{k+2}^{\infty} s^{-2} n_s r_s^{n_s} < C_1 \sum_{k+2}^{\infty} s^{-2} < C_1$$

if $r \in J_k$. C_1 as well as $C_2, \cdots, C_5$ denote positive numbers which do not depend on k.

Next, by (1),

(5) $$\left| R_k(\theta) - k^{-2} n_k e^{in_k\theta} \right| \leqq \sum_{s=1}^{k-1} s^{-2} n_s < (k-1) n_{k-1} < n_k^{1/2}.$$

In particular

(6) $$\left| R_k(\theta) \right| > C_2 k^{-2} n_k,$$

and

(7) $$\left| n_k\theta - \arg R_k(\theta) \right| < C_3 k^2 n_k^{-1/2} \qquad (\text{mod } 2\pi).$$

Now if $\theta \in T_k$, we have, for some $r \in J_k$,

(8) $$\begin{aligned} \left| R_k(\theta) + (k+1)^{-2} n_{k+1} (re^{i\theta})^{n_{k+1}} \right| &= \left| g(re^{i\theta}) + P_k(r,\theta) - Q_k(r,\theta) \right| \\ &< A + 1 + C_1 \end{aligned}$$

by (3) and (4). Hence (6) implies

(9) $$\left| n_{k+1}\theta - \pi - \arg R_k(\theta) \right| < C_4 k^2 n_k^{-1} \qquad (\text{mod } 2\pi).$$

By (7) and (9),

(10) $$\left| (n_{k+1} - n_k)\theta - \pi \right| < C_5 k^2 n_k^{-1/2} \qquad (\text{mod } 2\pi),$$

so that

$$m(T_k) < 2C_5 k^2 n_k^{-1/2}. \tag{11}$$

If $\theta \in E_A$, then θ is in infinitely many of the sets T_k, so that $E_A \subset \bigcup_{k=p}^{\infty} T_k$, for every positive integer p. Hence

$$m(E_A) \leqq \sum_{k=p}^{\infty} m(T_k). \tag{12}$$

By (11) and (1), the ratio test shows that $\sum m(T_k)$ converges. Letting $p \rightarrow \infty$ in (12), we see that $m(E_A) = 0$.

References

1. Otto Frostman, *Sur les produits de Blaschke*, Fysiogr. Sällsk. Lund Förh. vol. 12 (1942) pp. 169–182.

2. Rolf Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Paris, 1929.

University of Rochester