A NOTE ON FOURIER-STIELTJES TRANSFORMS¹

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Given a locally compact Abelian group G , let μ be a bounded Radon measure on the character group G^* . Then $\sum_{\nu \in G^*} |\mu\{y\}| < \infty$ —i.e. at most a countable set of points (y_n) are of nonzero μ -measure and $\sum_n |\mu\{y_n\}| < \infty$. μ thus admits the decomposition

$$\mu = \sum_n \mu\{y_n\} \mu_n + \nu,$$

where μ_n is a unit measure concentrated at the point y_n , and ν is a measure vanishing on single points (continuous measure). The Fourier transform of μ ,

$$\tilde{\mu}(x) = \int_{G^*} (x, -y) d\mu(y) = \sum_n \mu\{y_n\} (x, -y_n) + \int_{G^*} (x, -y) d\nu(y),$$

is a bounded uniformly continuous function defined on G —in fact, a weakly almost periodic (w.a.p.) function. (Cf. [1; 2]).² We show that some theorems of Wiener [7], Beurling and Helson [3], and Hewitt [4] are contained in the following corollary of the Parseval equation for w.a.p. functions:

$$\text{THEOREM 1. } M_x[|\tilde{\mu}(x)|^2] = \sum_{\nu \in G^*} |\mu\{y\}|^2.$$

Here the mean value $M(f) = M_x f(x)$ of a w.a.p. function f may be defined as the (necessarily unique) constant that is the uniform limit of convex combinations of translates of f . In particular: M is a linear functional. If f is almost periodic, $M(|f|) = 0$ only when $f = 0$. $M_x(x, y) = 0$ if $y \neq 0$, $= 1$ if $y = 0$. When G is the additive group of the reals (R_1), M has the representation

$$M(f) = \lim L^{-1} \int_0^L f(x) dx.$$

PROOF OF THEOREM 1. Given a w.a.p. function $f(x)$ on G with Fourier coefficients $a(y) = M_x[(x, y)f(x)]$, the Parseval equation reads:

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² Numbers in brackets refer to the bibliography at the end of the paper.

$$M[|f|^2] = \sum_{y \in G^*} |a(y)|^2.$$

When $f(x)$ has the form $\tilde{\mu}(x)$ it remains to establish:

LEMMA. $a(y) = \mu\{y\}$.

It is sufficient to show that $a(0) = M(\tilde{\mu}) = \mu\{0\}$, for the general case follows on translation of μ . When $G = R_1$, a standard Fubini and dominated convergence argument yields

$$M_x \int_{-\infty}^{\infty} (x, -y) d\mu(y) = \int_{-\infty}^{\infty} M_x(x, -y) d\mu(y) = \mu\{0\}.$$

The difficulty in general groups is that the mean value is represented as the limit of a directed set of convex combinations of translation operators rather than a sequence. However, it is clear that $M(\tilde{\mu}) = \mu\{0\}$ for a discrete measure μ ($\nu=0$), owing to uniform convergence of the series $\sum_n (x, -y_n) \mu\{y_n\}$. Hence we need only show that $M(\tilde{\nu})=0$ for a continuous measure ν .

Given any compact set C^* in G^* containing 0 as an interior point, write $\tilde{\nu}(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \int_{C^*} (x, -y) d\nu(y); \quad f_2(x) = \int_{G^* - C^*} (x, -y) d\nu(y).$$

Since $|M(f_1)| \leq \|f_1\|_{\infty} \leq |\nu|(C^*)$ is arbitrarily small for sufficiently small C^* , we need only show that $M(f_2)=0$. But it is well known (cf. [5, p. 146]) that one can find a function g in $L^1(G)$ such that $\tilde{g}(y) = \int_G (x, y) g(x) dx$ vanishes on $G^* - C^*$ and $\tilde{g}(0)=1$. Consider the convolution $(f_2 * g)(x) = \int_G f_2(x-t) g(t) dt = \int_{G^* - C^*} (x, -y) \tilde{g}(y) d\nu(y) = 0$. Since $M(f_2)$ is a uniform limit of convex combinations of translations $f_2(x-t)$ of $f_2(t)$,

$$0 = \int_G M(f_2) g(t) dt = M(f_2) \tilde{g}(0) = M(f_2).^* \text{ q.e.d.}$$

Theorem 1 for the real line is the intersection of several theorems of Wiener [7]. An immediate corollary is

THEOREM 2 (BEURLING-HELSON). If $|\tilde{\mu}(x)| \equiv 1$, then

$$\sum_{y \in G^*} |\mu\{y\}|^2 = 1.$$

^{*} In effect we have shown that 0 is not in the "point" spectrum of f_2 , but an obvious modification of the argument shows that 0 lies in no part of the spectrum of f_2 (cf. [2]).

Suppose that $\tilde{\mu}(x)$ is almost periodic. Then the difference $\tilde{\nu}(x)$ is an almost periodic function all of whose Fourier coefficients $a(y)$ vanish. Then $M(|\tilde{\nu}|^2) = 0$, whence $\tilde{\nu} = 0$, and then $\nu = 0$ by the uniqueness theorem for Fourier-Stieltjes transforms. We thus obtain a result established by Segal [6] (and extended by Hewitt [4]) only for a special class of groups in which the mean value has a sequential limit representation:

THEOREM 3. *If $\tilde{\mu}(x)$ is almost periodic, μ is discrete ($\nu = 0$).*

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