

*Then, for  $G \simeq K$  it is necessary and sufficient that  $R$  be cyclically equivalent to  $A^n(a_1, a_2)$  where  $A(a_1, a_2)$  is a primitive element in the free group  $F(a_1, a_2)$ .*

#### REFERENCES

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## A THEOREM ON COMMUTATIVE POWER ASSOCIATIVE LOOP ALGEBRAS<sup>1</sup>

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Let  $L$  be a loop, written multiplicatively, and  $F$  an arbitrary field. Define multiplication in the vector space  $A$ , of all formal sums of a finite number of elements in  $L$  with coefficients in  $F$ , by the use of both distributive laws and the definition of multiplication in  $L$ . The resulting *loop algebra*  $A(L)$  over  $F$  is a linear nonassociative algebra (associative, if and only if  $L$  is a group).

An algebra  $A$  is said to be power associative if the subalgebra  $F[x]$  generated by an element  $x$  is an associative algebra for every  $x$  of  $A$ .

**THEOREM.** *Let  $A(L)$  be a loop algebra over a field of characteristic not 2. A necessary and sufficient condition that  $A(L)$  be a commutative, power associative algebra is that  $L$  be a commutative group.*

**PROOF.** Assume that  $A(L)$  is a commutative, power associative algebra. Clearly  $L$  must be commutative and  $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x$  for all  $x$  of  $A(L)$ . Under the hypothesis that the characteristic of  $F$  is not 2, a linearization<sup>2</sup> of this power identity yields

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<sup>2</sup> See A. A. Albert, *On the power associativity of rings*, Summa Brasiliensis Mathematicae vol. 2, no. 2, pp. 21-32.

$$(1) \quad \sum_6 (xy + yx)(zw + wz) = \sum_4 \left[ \sum_3 (zw + wz)y \right] x,$$

for all  $x, y, z, w$  of  $A(L)$ . The sums in (1) are taken over all possible selections of the symbols involved and  $\sum_k$  is the sum of  $k$  terms. We now restrict our attention to elements of  $L$  in  $A(L)$  and set  $w=z$  in (1). Thus we obtain,

$$(2) \quad 4z^2(xy) + 8(xz \cdot zy) = (z^2y)x + (z^2x)y + 2(yz \cdot z)x + 2(xz \cdot z)y \\ + 2(zy \cdot x)z + 2(zx \cdot y)z + 2(xy \cdot z)z,$$

as a necessary condition for the power associativity of  $A(L)$ .

The left member of (2) consists of at most two distinct elements of  $L$ . Hence the resolution of the right member into two distinct elements of  $L$  requires that two of the three elements  $(zy \cdot x)z$ ,  $(zx \cdot y)z$ ,  $(zy \cdot z)z$  be equal. This, together with the commutativity of  $L$ , implies that  $(xy)z = x(yz)$  for all  $x, y, z$  of  $L$ .

The proof of the sufficiency is obvious.

The situation is expectedly different if the characteristic of  $F$  is two. For example, let  $L$  be a commutative loop such that  $x^2=1$  (1 the identity element of  $L$ ) for all  $x$  of  $L$ . Then if  $\alpha = \sum_{i=1}^n a_i x_i$ , where  $a_i \in F$  and  $x_1=1$ ,

$$\alpha^2 = \left[ \sum_{i=1}^n a_i^2 \right] x_1;$$

hence, the power associativity of the loop algebra  $A(L)$  is a trivial verification.

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