THE SPLITTING OF CERTAIN SOLVABLE GROUPS[1]

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Let G be a finite group. We shall designate the commutator subgroup of G by $G^2=[G, G]$; this is the group generated by all commutators $[g, h]=ghg^{-1}h^{-1}$. Inductively $G^n=[G^{n-1}, G]$ is defined to be the group generated by commutators of elements of G with elements of G^{n-1}; and G^* will designate $\bigcap_{n=1}^{\infty}G^n$. It should be recalled that G is nilpotent if $G^*=E$, the subgroup consisting of the identity element, or equivalently, if G is the direct product of p-groups.

Our object here is to show that when G^* is Abelian then there is a nilpotent group X so that $G=XG^*$ where $X\cap G^*=E$. If there are two such splittings of G into XG^* and YG^* then Y and X are conjugates by an element of G^*. If x is in the center of X then x does not commute with any of its conjugates. As a consequence of the properties of the splitting it will follow that if G has no center and G^* is Abelian, then both G and its group of automorphisms are contained in the holomorph of G^*.

We shall also give an example to show that the hypothesis that G^* be nilpotent instead of Abelian is insufficient to insure a splitting of G in this fashion.

The splitting of G. In order to show the existence of the splitting mentioned above we first prove the following fact.

LEMMA. *If G/G^* is cyclic, that is, if G is generated by G^* and an element x, and if G^* is Abelian, then every element of G^* is of the form $[x, k]$ for some $k\in G^*$. Thus the map sending k into $[x, k]$ is a* 1-1 *map of G^* onto itself.*

PROOF. To prove this we shall use the following easily verified rules for commutators (cf. [**2**, p. 60]):

$$[a, bc] = [a, b][a, c]^b \quad \text{where } g^b \text{ denotes } bgb^{-1},$$
$$[ab, c] = [b, c]^a[a, c],$$

and

$$[a, b] = [b, a]^{-1}.$$

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Then remembering that G^* is an Abelian normal subgroup and that $G^2=G^*$ we have for g, h in G^*

$$(1) \qquad [x, gh] = [x, g][x, h]^g = [x, g][x, h]$$

and therefore also

$$(2) \quad [x, g^{-1}] = [x, g]^{-1} = [g, x] \text{ since } e = [x, g^{-1}g] = [x, g^{-1}][x, g].$$

Now the elements of G are of the form gx^r, hx^s for r and s integers and hence $[gx^r, hx^s]=[gx^r, h][gx^r, x^s]=[x^r, h][g, x^s]$. But $[x^r, h] =[x^{r-1}, h]^x[x, h]=[x^{r-1}, h^x][x, h]$ and therefore by an induction argument $[x^r, h]=[x, \bar{h}]$ for some $\bar{h}$ in G^*. Also $[g, x^s]=[x^s, g]^{-1}$ and therefore $[g, x^s]=[x, \bar{g}]$ for $\bar{g}$ in G^*. It follows that every commutator and hence in view of (1) every element of G^2 is of the form $[x, k]$ for some k in G^* as the lemma asserts. That the map sending k into $[x, k]$ is a 1-1 map of G^* onto itself follows readily from this.

COROLLARY. *If H is a normal subgroup of G contained in G^* then $[x, H]=H$ where $[x, H]$ denotes the set of commutators $[x, h]$ for $h\in H$. If K is the group generated by x and H then K is not nilpotent and in fact $K^*=H$.*

We can now prove the splitting theorem.

THEOREM 1. *If G is a finite group so that G^* is Abelian then G contains a proper subgroup X such that $G^*\cap X=E$, $G=G^*X$, and consequently X is isomorphic to G/G^* and is nilpotent.*

PROOF. G^* is normal in G. We shall first consider the case where G^* is minimal normal in G, that is G^* does not properly contain any normal subgroup of G other than E. Since G is not nilpotent the Φ subgroup of G (cf. [2, p. 114]) does not contain G^2. Therefore there is a minimal set of generators of G, $g_1, \cdots, g_k$, where at least one of the generators, say g_k, is in G^2. Then $g_1, \cdots, g_{k-1}$ generate a proper subgroup K of G. Since G/G^* is nilpotent, $g_1G^*, \cdots, g_{k-1}G^*$ generate G/G^* (cf. [2, p. 114] again) and $G=G^*K$. Then $K\cap G^*$ is normal in K and in G^*, hence in G. Since K is a proper subgroup, $K\cap G^*$ must be E and the theorem is proved when G^* is minimal normal.

If G^* is not minimal normal then we are going to show the existence of a subgroup H properly contained in G^* such that $[G, H]=H$. This is clearly true if G^* has order not a power of a prime; hence suppose G^* has order a power of a prime p. Since G/G^* is a direct product of p-groups, G has a normal non-nilpotent (cf. [1, pp. 98–102]) subgroup Q containing G^* so that Q/G^* has order a power of a prime $q\neq p$. Hence there is an element of q power order not in the centralizer Z of G^*. Since Z is normal in G and G/Z is the direct product of p-groups, there is a central element of G/Z of order a power of q and consequently a normal subgroup K of G generated by Z and an element x of order a power of q. x does not commute with all the elements of G^*; therefore K is not nilpotent and we have $E \neq K^* \subset G^*$, K^* normal in G. The elements of K are of the form $x^r z$ for r integral and z in Z. Therefore if $g \in G^*$, $[x^r z, g] = [x^r, g]$ and we see that if L is the group generated by x and K^*, then $L^* = K^*$. Now we can apply the corollary to the lemma to see that if H is any normal subgroup contained in K^* then $[x, H] = H$.

Now if $K^* \neq G^*$, then K^* is the desired subgroup such that $[G, K^*] = K^*$. If $K^* = G^*$ then any normal subgroup H of G contained in G^* has the property that $[G, H] = H$ since $[x, H] = H$. In either event we can proceed by induction to finish the proof of the theorem. For let $H \neq E$ be properly contained in G^* such that $[G, H] = H$. Then by an induction argument G has a proper subgroup K so that $G/H = K/H \cdot G^*/H$ or $G = KG^*$ with $K \cap G^* \subset H$. Then $[K, H] = H$ since $[G, H] = H$ and $K^* \subset G^* \cap K \subset H$; hence $K^* = H \neq E$ and by the induction argument $K = XK^*$ where $X \cap K^* = E$. Finally $G = KG^* = XG^*$ and $X \cap G^* \subset K \cap G^* \subset H$; hence $X \cap G^* \subset H \cap X = K^* \cap X = E$ and the theorem is proved.

Remark. We shall give here an example to show that the above type of splitting is in general impossible when G^* is nilpotent even if G/G^* is Abelian. For p a prime not 2 let H be a group of order p^4, generated by elements a, b, and c; a and b of order p, c of order p^2, and $c^p = [a, b]$, $[c, a] = [c, b] = e$, the identity. Let h be an automorphism of H sending a into a^{-1}, b into b^{-1}, and c into c; and let G be the holomorph of H with h of order $2p^4$. Then G^* consists of the group of order p^3 generated by a and b. Since c is of order p^2 and $c^p = [a, b]$ the impossibility of a splitting as in the theorem is clear.

On the conjugacy of the complements of G^*. If $G = AB$ where A and B are subgroups whose intersection is the identity we shall call A a complement of B in G. Our main result here is then the following.

Theorem 2. *If G^* is Abelian and if X and Y are two complements of G^*, then for some $h \in G^*$, $X = hYh^{-1}$.*

Proof. First suppose that G^* is a minimal normal subgroup of G; then G^* has order a power of some prime p. Let x be of order q prime to p in the center of X. If x is not in the centralizer of G^* then x and G^* generate a normal subgroup R of G which is not nilpotent and therefore $R^* = G^*$ by the minimality condition on G^*. It follows from the corollary to the lemma of the last section that every element h of G^* is of the form $xgx^{-1}g^{-1}$ for some g in G^*.

Now there is a $y \neq e$ in $R \cap Y$ such that $y = x^{-1}h$ and therefore $y = gx^{-1}g^{-1}$. Suppose $Y \neq gXg^{-1}$; then since $G = G^*(gXg^{-1})$ there is a k in Y so that $k = hgmg^{-1}$ for some h in G^*, $h \neq e$, and m in X. But then since y is a conjugate of x^{-1}, x in the center of X, it follows that $[k, y] = [hgmg^{-1}, y] = [h, y]$; hence $[[k, y] \cdots y] = [[h, y] \cdots y] \neq e$ since $[y, G^*] = G^*$. But this is a contradiction of the nilpotency of Y and we conclude that $Y = gXg^{-1}$ when x is not in the centralizer of G^*.

If x is in the centralizer of G^* then x is in the center of G and since x is in every Sylow q group of G, x is in Y. Then by an induction argument the theorem is true in $G/(x)$ and from this the theorem follows for G when G^* is minimal normal.

The general case now follows easily from this. As in the proof of Theorem 1 there is an x and a normal subgroup H of G properly contained in G^* such that $[x, H] = H$. Without loss in generality x can be taken in X. By an induction argument, $Y/H = (gH)X/H(gH)^{-1}$ for some gH in G^*/H and hence if g is an element of gH then gXg^{-1} is a complement of K^* in the proper subgroup K generated by Y and H. But $[K, H] = H = K^*$ and by the induction assumption again there is an $h \in H$ so that $h(gXg^{-1})h^{-1} = Y$. This completes the proof of the theorem.

REMARK. If h is in G^* then either h is in the center of G or $hXh^{-1} \neq X$. For if $hXh^{-1} = X$ then X is normal in the group generated by X and h; so also is (h). Hence $[x, h] \subset X \cap (h) = E$ and h is in the center of G.

REMARK. If x is in the center of X then x does not commute with any of its conjugates. For if $y \neq x$ is a conjugate of x, then it is clear that $y = hxh^{-1}$ for some $h \in G^*$. If x and y commute, then x commutes with $[h, x] \in G^*$ where $[h, x] \neq e$ since $y \neq x$. Hence $[x, G^*] \neq G^*$. Let K be the group generated by x and G^*; then K^* is properly contained in G^* and, being normal in G, K contains a normal subgroup H of G so that either $[H, x] = H$ or $[H, x] = E$. Then by an induction argument assuming the statement true in G/H, we see that since $y = hxh^{-1}$ then h must be in H and consequently $[h, x]$ is in H. Since $[h, x] \neq e$ and commutes with x it is not possible that $[H, x] = H$. On the other hand $[H, x]$ cannot be E for then $[h, x]$ would be e. We thus get a contradiction by assuming that x can commute with one of its conjugates.

On the group of automorphisms of G when G^* is Abelian and G has no center.

THEOREM 3. *If G is a group with no center and G^* is Abelian, then both G and A, the group of automorphisms of G, are contained in the holomorph R of G^*. Furthermore if D and F are complements of G^* in A, then there is an $h\in G^*$ so that $hDh^{-1}=F$.*

PROOF. That G is in R follows from Theorem 1; for $G=G^*X$ where G^* is normal in G and $G\cap X=E$.

Now G^* is a characteristic subgroup of G and therefore every automorphism of G maps G^* into itself. Suppose m is an automorphism of G which commutes with all the elements of G^*. Let L be the holomorph of G and m and let Z be the centralizer of G^* in L. Then Z is normal in L and hence for every $h\in G^*$, $[mh, G]\subset Z\cap G=G^*$. In view of Theorem 2 there is an h_0 in G^* so that mh_0 maps X into itself; that is, $[mh_0, X]\subset X$. Therefore $[mh_0, X]\subset X\cap G^*=E$ and mh_0 commutes with every element of X as well as of G^*; that is, mh_0 is the identity automorphism in G or m is the same as the inner automorphism determined by h_0^{-1}. It follows that A/G^* is isomorphic to a subgroup M of automorphisms of G^*, M containing X as a normal subgroup. Then N, the holomorph of M with G^* contains a copy of G and the centralizer of this copy of G is E. Then since N and A are both groups of automorphisms of G having the same order, A is isomorphic to N and hence A is a subgroup of R as the theorem asserts.

We now show that if D and F are two complements of G^* in A then there is an $h\in G^*$ so that $F=hDh^{-1}$. It is clear that $D\cap G$ is normal in D and is a complement of G^* in G; hence there is an $h\in G^*$ so that $h(D\cap G)h^{-1}=F\cap G$. But $F\cap G$ determines F completely; for if $F\cap G$ were normal in F and also in $F'\neq F$, then $F\cap G$ would be normal in the group generated by F and F' which must necessarily intersect G^* in a subgroup $Z\neq E$. But then $[Z, F\cap G]\subset(F\cap G)$ and $[Z, F\cap G]\subset G^*$, whence it follows that $[Z, F\cap G]=E$ and G has a nontrivial center contrary to hypothesis. It follows that $F\cap G$ determines F completely, and then hDh^{-1} must be F. This completes the proof of the theorem.

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