

AN INEQUALITY FOR LINEAR OPERATORS BETWEEN L^p SPACES

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1. Introduction. Let R and S be sets of points with completely additive non-negative measure ϕ and ψ defined over σ -rings $\mathcal{J}(R)$, $\mathcal{J}(S)$ of subsets of R and S respectively. Assume also that R and S are σ -finite with respect to ϕ and ψ . For any real number p , $1 \leq p < \infty$, $L^p(R, \phi)$ will denote the Banach space of real-valued functions $x(\psi)$ for which $\int_R |x(t)|^p d\phi < \infty$ where the norm is defined by $\|x\| = (\int_R |x(t)|^p d\phi)^{1/p}$. $L^q(S, \psi)$ will denote a similar space for the set S with measure ψ . It is known [2; 3] that the space $L^p(R, \phi)$ is equivalent as a Banach space to the space $V^p(R, \phi)$ of completely additive set functions of bounded p -variation where $p\text{-var } F(e) = \sup_{\pi} (\sum_{i=1}^n |F(e_i)|^p / [\phi(e_i)]^{p-1})^{1/p}$ where the supremum is taken with respect to all finite families π of disjoint subsets of R of finite nonzero measure.

If T is a bounded linear operator between two spaces of the above type, $L^p(R, \phi)$ and $L^q(S, \psi)$, it can be shown that T can be represented as the derivative of an integral involving a kernel. Thus $Tx = (d/de) \int_R K(e, t)x(t)d\phi$ where $K(e, t)$ is defined for $e \in F(R)$, $t \in R$, $\int_R K(e, t)x(t)d\phi \in V^q(S, \psi)$ and the symbol d/de applied to a completely additive and absolutely continuous set function denotes the integrable point function associated with it by the Radon-Nikodym theorem. This representation can be obtained by using the standard theorems for representations of linear functionals over spaces of the type $L^p(R, \phi)$ [1].

The problem in connection with this representation which presents a greater difficulty is that of determining necessary and sufficient conditions on the kernel to insure that the operator is one of the desired type and also to find a suitable expression or suitable inequalities for the norm of the operator T in terms of the kernel. Thus if the norm of T lies between two fixed multiples of some expression involving the kernel, this yields a convenient necessary and sufficient condition that T be bounded. In the case in which $1 \leq p < \infty$, $q=1$, such bounds were found on the norm of T and in case T was a positive operator, an exact expression for the norm of T was found by the author [2]. In this paper, the same techniques as were used in [2] are applied to the case for $q>1$ to determine a convenient expression for a lower bound on the norm of T under certain

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conditions on p and q and on the measure of S . The result is weaker than the result in the case $q=1$ but it does appear to give a non-trivial lower bound to the norm of the operator T .

2. Inequalities for the norm of the operator.

THEOREM. *Let T be a bounded linear operator from $L^p(R, \phi)$ to $L^q(S, \psi)$, $1 \leq p, q < \infty$. Then there exists a real-valued function $K(e, t)$ defined on $\mathcal{F}(S) \times R$ such that for each $x \in L^p(R, \phi)$,*

$$Tx = (d/de) \int_R K(e, t) x(t) d\phi$$

where $K(e, t)$ satisfies the following conditions:

(1) $\int_E K(\cdot, t) d\phi \in V^q(S, \psi)$ for each $E \in \mathcal{F}(R)$, $\phi(E) < \infty$.

(2) $K(e, \cdot) \in L^{p'}(R, \phi)$ for each $e \in \mathcal{F}(S)$, $1/p + 1/p' = 1$.

(3) $\|T\| \leq q\text{-var}_S \left[\left(\int_R |K(e, t)|^{p'} d\phi \right)^{1/p'} \right]$.

(4) If $p \leq q$ and $\psi(S)$ is finite then the integral $\int_R |K(e, t)|^{p'} d\phi$ has bounded q -variation.

(5) $(q\text{-var}_S \int_R |K(e, t)|^{p'} d\phi)^{1/p'} / \sup_{\pi} \left(\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)/qp'} \right) \leq \|T\|$ where the supremum is taken over all finite disjoint families of sets of S of finite nonzero measure.

In particular, if $\psi(S) \leq 1$ and $p \leq q$ then

$$\left[q\text{-var}_S \int_R |K(e, t)|^{p'} d\phi \right]^{1/p'} \leq \|T\|.$$

PROOF. If T is bounded linear from $L^p(R, \phi)$ to $L^q(S, \psi)$ and if $Tx=y$, let the operator U be defined from $L^p(R, \phi)$ to $V^q(S, \psi)$ as $Ux = \int_S y(s) d\psi$. U is bounded linear from $L^p(R, \phi)$ to $V^q(S, \psi)$ and has the same norm as T [2; 3] and hence for any fixed $e \in F(S)$, the functional defined by assigning to each $x \in L^p(R, \phi)$ the value of Ux at e yields a linear functional over $L^p(R, \phi)$. Thus, by using the known representation theorems for linear functionals on L^p , [1], $Ux(e) = \int_R K(e, t) x(t) d\phi$ where $K(e, \cdot) \in L^{p'}(R, \phi)$ for each $e \in F(S)$. Thus $Tx = (d/de) \int_R K(e, t) x(t) d\phi$ and (2) is proved. If $\gamma_E(t)$ is the characteristic function of $E \in \mathcal{F}(S)$, then $\gamma_E \in L^p(R, \phi)$ and $\int_R K(e, t) \gamma_E(t) d\phi = \int_E K(e, t) d\phi \in V^q(S, \psi)$. Thus (1) is true. The Hölder inequality can be used to prove (3). For $x \in L^p(R, \phi)$,

$$\begin{aligned} \|Tx\| &= \|Ux\| = q\text{-var}_S \int_R K(e, t) x(t) d\phi \\ &\leq q\text{-var}_S \left(\int_R |K(e, t)|^{p'} d\phi \right)^{1/p'} \left(\int_R |x(t)|^p d\phi \right)^{1/p}. \end{aligned}$$

The principal part of the proof of the rest of the theorem lies in the proof of the inequality (5). Let $\Pi = \{\pi\}$ be the class of all families of disjoint measurable subsets $\pi = \{E_i\}$ of finite nonzero measure in R . It was proved in [2] that the set $B \subset L^p(R, \phi)$ consisting of all functions of the type $z(t) = \sum_{i=1}^n \{[a_i \gamma_{E_i}(t)] / [\psi(E_i)]^{1/p}\}$ is dense in the unit sphere of $L^p(R, \phi)$ where $\gamma_E(t)$ denotes the characteristic function of E , $\sum_{i=1}^n |a_i|^p \leq 1$, and $\{E_i\}$ ranges over all of Π . Thus $\|T\| = \sup_{z \in B} \|Tz\|$. Let $\{e_j\}$ be any finite family of sets in $\mathcal{J}(S)$ of finite nonzero measure. Then

$$\begin{aligned} & \left\{ \sum_{j=1}^m \frac{\left| \int_R K(e_j, t) z(t) d\phi \right|^q}{[\psi(e_j)]^{q-1}} \right\}^{1/q} \\ &= \left\{ \sum_{j=1}^m \frac{\left| \sum_{i=1}^n a_i \left(\int_{E_i} K(e_j, t) d\phi \right) / [\phi(E_i)]^{1/p} \right|^q}{[\psi(e_j)]^{q-1}} \right\}^{1/q} \\ &= \left\{ \sum_{j=1}^m \left| \sum_{i=1}^n a_i \frac{\int_{E_i} K(e_j, t) d\phi}{[\psi(e_j)]^{1/q'} [\phi(E_i)]^{1/p}} \right|^q \right\}^{1/q} \leq \|T\|. \end{aligned}$$

In particular, for any $e \in F(S)$, $0 < \psi(e) < \infty$,

$$\sum_{i=1}^n a_i \frac{\int_{E_i} K(e, t) d\phi}{[\phi(E_i)]^{1/p}} \leq \|T\| [\psi(e)]^{1/q'}$$

for all families $\{E_i\}$, $i=1, 2, \dots, n$, as described above and for all finite sequences $\{a_i\}$, $i=1, 2, \dots, n$, of real numbers such that $\sum_{i=1}^n |a_i|^p \leq 1$. Let $\beta_i = [\int_{E_i} K(e, t) d\phi] / [\phi(E_i)]^{1/p}$, $i=1, 2, \dots, n$. This sequence $\{\beta_i\}$ can be considered as an infinite sequence by adding zeros after the n th term. Hence, if $\{a_i\}$ is any sequence for which $\sum_{i=1}^n |a_i|^p \leq 1$, $|\sum_{i=1}^n \beta_i a_i| \leq \|T\| [\psi(e)]^{1/q'}$. Hence the sequence $\{\beta_i\}$ represents a linear functional f over l^p of norm not exceeding $\|T\| [\psi(e)]^{1/q'}$ and by using the known representation theorems for the norm of such a functional, $\|f\| = \sup_{\sum |a_i| \leq 1} |\sum_{i=1}^\infty \beta_i a_i| = (\sum_{i=1}^\infty |\beta_i|^{p'})^{1/p'} \leq \|T\| [\psi(e)]^{1/q'}$. Hence

$$\left(\sum_{i=1}^n \left| \int_{E_i} K(e, t) d\phi \right| / [\phi(E_i)]^{1/p} \right)^{p'} \leq \|T\| [\psi(e)]^{1/q'}$$

for all partitions $\{E_i\}$ of R as described above and for any $e \in F(S)$, $0 < \psi(e) < \infty$. Hence if the supremum is taken over all such partitions

$$\begin{aligned}
\sup_{\pi} \left\{ \sum_{i=1}^n \left| \frac{\int_{E_i} K(e, t) d\phi}{[\phi(E_i)]^{1/p}} \right|^{p'} \right\}^{1/p'} &= \sup_{\pi} \left\{ \sum_{i=1}^n \frac{\left| \int_{E_i} K(e, t) d\phi \right|^{p'}}{[\phi(E_i)]^{p'-1}} \right\}^{1/p'} \\
&= p'\text{-var}_R \left| \int_E K(e, t) d\phi \right| \\
&= \left(\int_R |K(e, t)|^{p'} d\phi \right)^{1/p'} \\
&\leq \|T\| [\psi(e)]^{(q-1)/q}
\end{aligned}$$

and hence for any $e \in \mathcal{F}(S)$ of finite nonzero measure,

$$\frac{\left[\int_R |K(e, t)|^{p'} d\phi \right]^q}{[\psi(e)]^{q-1}} \leq \|T\|^{p'q} [\psi(e)]^{(p'-1)(q-1)}$$

Let $\{e_j\}$ be any finite family of subsets of S of finite nonzero measure, $j=1, 2, \dots, m$. Then

$$\begin{aligned}
\sum_{j=1}^m \frac{\left[\int_R |K(e_j, t)|^{p'} d\phi \right]^q}{[\psi(e_j)]^{q-1}} &\leq \|T\|^{p'q} \left(\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} \right), \\
\left(\sum_{j=1}^m \frac{\left[\int_R |K(e_j, t)|^{p'} d\phi \right]^q}{[\psi(e_j)]^{q-1}} \right)^{1/q} &\leq \|T\|^{p'} \left(\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/q}.
\end{aligned}$$

If the least upper bound of the two quantities is taken with respect to all such families $\{e_j\}$ of S then

$$\begin{aligned}
\sup_{\pi} \left(\sum_{j=1}^m \frac{\left[\int_R |K(e_j, t)|^{p'} d\phi \right]^q}{[\psi(e_j)]^{q-1}} \right)^{1/q} \\
= q\text{-var}_{\pi} \left(\int_R |K(e, t)|^{p'} d\phi \right) \leq \|T\|^{p'} \sup_{\pi} \left(\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/q}
\end{aligned}$$

and hence

$$\left(q\text{-var}_{\pi} \int_R |K(e, t)|^{p'} d\phi \right)^{1/p'} / \sup_{\pi} \left(\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/q p'} \leq \|T\|.$$

If S has finite measure and $(p'-1)(q-1) \geq 1$, the supremum in the

denominator is finite and this implies that $\int_R |K(e, t)|^{p'} d\phi$ has bounded q -variation. In particular, if the measure of S is less than one and $(p'-1)(q-1) \geq 1$, then for each partition $\{e_j\}$, $\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} \leq 1$ and

$$\left[q\text{-var}_S \left(\int_R |K(e, t)|^{p'} d\phi \right) \right]^{1/p'} \leq \|T\|.$$

However, $(p'-1)(q-1) \geq 1$ implies $q \geq p'/(p'-1) = p$. Thus if $q \geq p$ the above inequalities will hold. Hence if $\psi(S) \leq 1$, $q \geq p$, the double inequality $[q\text{-var} \int_R |K(e, t)|^{p'} d\phi]^{1/p'} \leq \|T\| \leq q\text{-var} (\int_R |K(e, t)|^{p'} d\phi)^{1/p'}$ will hold. Thus if the right member of the inequality is finite the integral represents a bounded linear operator between L^p and V^q , and if the norm of T is finite, the left member of the inequality is finite. It is to be noted that since there is no guarantee that $\int_R |K(e, t)|^{p'} d\phi$ is an additive set function, the theorem of Riesz on the equivalence of the q -variation to the q th integral norm will not apply and the q -variation cannot be replaced by an integral.

In case $p=q$, then $(q-1)(p'-1)=1$ and in this case

$$\sup_{\pi} \sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} = \psi(S).$$

Hence we have the

COROLLARY. If S has finite measure and $p=q$, then

$$\left(\left[q\text{-var}_S \int_R |K(e, t)|^q d\phi \right] / [\psi(S)]^{1/q} \right)^{1/q'} \leq \|T\|.$$

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