A NOTE ON A THEOREM OF K. G. WOLFSON

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1. In [3] K. G. Wolfson proves the following:

THEOREM. *A commutative B^*-algebra K containing an identity e (with $\|k^* \cdot k\| = \|k\|^2$ for all $k \in K$ and $\|e\| = 1$) is isomorphic (in a norm and * preserving manner) to an algebra $B(X)$ of all bounded complex-valued functions on an essentially unique set X if and only if:*

(1) *Every nonzero closed ideal of K contains a minimal ideal;*

(2) *The sum of two annulets is an annulet.*

An annulet is here understood to be an ideal I of K with which is associated a subset $G \subseteq K$ such that

$$I = [k \in K \mid k \cdot g = 0 \text{ for all } g \in G].$$

The two characterizing traits therein presented neither are self-evident in $B(X)$ nor touch upon the projections and Gelfand-Neumark [1; 2] continuous function representations so commonly used in the analysis of B^*-algebras. It is with this in mind that the following characterizations are offered:

THEOREM 1. *A necessary and sufficient condition that a B^*-algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff space X be isomorphic (in a norm and * preserving manner) to a B^*-algebra $B(\overline{X}_0)$ of all bounded complex-valued functions on a set $\overline{X}_0$ (essentially unique) is that the compact space X contain a dense subset X_0 of points each of which is an open-closed subset of X and such that each nonempty subset of points in X_0 is contained in an open-closed subset of X which includes of X_0 just that subset.*

PROOF. *Sufficiency.* Assume that the compact Hausdorff space X contains a subset X_0 as described above. Viewing X_0 simply as a set, form the B^*-algebra $B(X_0)$ and consider the map $C(X) \to B(X_0)$ under which c in $C(X)$ becomes $\bar{c}$ in $B(X_0)$ with $\bar{c}(x_0) = c(x_0)$ for all x_0 in X_0. This map is linear and * preserving. Since X_0 is dense in X the map is norm preserving and thus 1-1. Finally this map is onto: thus let $b \in B(X_0)$ be arbitrary and assume that its range as x_0 varies over X_0 falls in the interior of a right-open square of side $2M$ about the origin in the complex plane. For each positive integer n the de-

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composition of this square into $2^n \times 2^n$ equal right-open squares determines a division of X_0 into $2^n \times 2^n$ disjoint subsets (some perhaps empty) and thus a decomposition of X into $2^n \times 2^n$ disjoint open-closed subsets (some perhaps the empty set) X_i, $i=1, \cdots, 2^n \times 2^n$. If λ_i is the center of the ith of the above squares and $\chi(X_i)$ is the characteristic function of the corresponding open-closed subset of X, then $c_n(x) = \sum_{i=1}^{2^n \times 2^n} \lambda_i \cdot \chi(X_i)[x]$ is an element of $C(X)$ such that $|\bar{c}_n(x_0) - b(x_0)| \leq M/n2^{1/2}$ for all x_0 in X_0. Thus $\{\bar{c}_n\}$ is a Cauchy sequence in $B(X_0)$ and, since norms are preserved, $\{c_n\}$ is a Cauchy sequence in $C(X)$ which converges point-wise (and uniformly) to an element $c \in C(X)$. Finally, for each x_0 in X_0:

$$c(x_0) = \lim_n c_n(x_0) = \lim_n \bar{c}_n(x_0) = b(x_0).$$

Necessity. Assume now that $C(X)$ is isomorphic as a B^*-algebra to some $B(\overline{X}_0)$. In $B(\overline{X}_0)$ denote by $\bar{e}_{\bar{x}_0}$, $\bar{x}_0 \in \overline{X}_0$, the system of all characteristic functions of a single point $\bar{x}_0 \in \overline{X}_0$ and by $\bar{e}_{\overline{A}}$, $\overline{A} \in S(\overline{X}_0)$, the system of all characteristic functions on the various subsets $\overline{A} \in S(\overline{X}_0)$ of $\overline{X}_0$. Denote by $e_{\bar{x}_0}$ and $e_{\overline{A}}$ the elements of $C(X)$ which correspond under the assumed isomorphism to $\bar{e}_{\bar{x}_0}$ and $\bar{e}_{\overline{A}}$ in $B(\overline{X}_0)$. It is clear that $e_{\bar{x}_0}$ and $e_{\overline{A}}$ are idempotent, hermitian in $C(X)$ and thus are the characteristic functions of open-closed subsets of X. For each $c \in C(X)$ and each $e_{\bar{x}_0}$, $c \cdot e_{\bar{x}_0} = \lambda e_{\bar{x}_0}$ where λ is a complex scalar, since the corresponding multiplicative property clearly holds in $B(\overline{X}_0)$. From this it follows, by use of the theorem that for any open set O in compact X and any point x_0 in O there is an element c in $C(X)$ with $c(x_0)=1$ and $c(x)=0$ for $x \in O$, that $e_{\bar{x}_0}$ is the characteristic function of a single point $x_0 \in X$. Similarly for each $c \neq 0$ in $C(X)$ there exists at least one $e_{\bar{x}_0}$ in $C(X)$ such that $c \cdot e_{\bar{x}_0} \neq 0$ since the corresponding property holds in $B(\overline{X}_0)$. From this it follows by the same theorem that the points x_0 determined by the $e_{\bar{x}_0}$ are dense in X. Finally for any collection of the x_0 in X the open-closed subset A of X determined by the idempotent, hermitian element $e_{\overline{A}}$ of $C(X)$ corresponding to the element $\bar{e}_{\overline{A}}$ of $B(\overline{X}_0)$, where $\overline{A}$ contains all and only the corresponding $\bar{x}_0$, itself contains all and only the given x_0 since $e_{\bar{x}_0} \cdot e_{\overline{A}} = e_{\bar{x}_0}$ if and only if $\bar{e}_{\bar{x}_0} \cdot \bar{e}_{\overline{A}} = \bar{e}_{\bar{x}_0}$.

Since a dense subset in X of points that are open-closed subsets necessarily includes all such points, the essential identity of the sets $\overline{X}_0$, X_0 is clear.

2. Let K be an arbitrary commutative B^*-algebra with unit e, $\|e\|=1$, and with $\|k^* \cdot k\| = \|k\|^2$ for all $k \in K$. In the collection of all projections (idempotent, hermitian, nonzero elements) of K distin-

guish the collection (possibly empty) e_{x_0}, $x_0 \in X_0$, of projections such that for each $k \in K$ and each e_{x_0}, $x_0 \in X_0$, there exists a complex scalar λ such that $k \cdot e_{x_0} = \lambda e_{x_0}$. Such projections may be called minimal projections. Here the indexing set X_0 is assumed so chosen that $x_0 \neq y_0$ in X_0 implies $e_{x_0} \neq e_{y_0}$ in K.

THEOREM 2. *A necessary and sufficient condition that a commutative B^*-algebra K with identity e and with $\|k^* \cdot k\| = \|k\|^2$ for all k be isomorphic (in a norm and * preserving manner) to a B^*-algebra $B(\overline{X}_0)$ of all complex-valued functions on a set $\overline{X}_0$ is that the subset of all minimal projections e_{x_0}, $x_0 \in X_0$, of K be such that:*

(1) *for each $k \neq 0$ in K there exists at least one minimal projection e_{x_0}, $x_0 \in X_0$, such that $k \cdot e_{x_0} \neq 0$.*

(2) *for each subcollection $A \subseteq X_0$ of minimal projections there exists in K a projection e_A such that $e_A \cdot e_{x_0} = e_{x_0}$ for $x_0 \in A$ and $e_A \cdot e_{x_0} = 0$ for $x_0 \notin A$.*

When these conditions are satisfied the indexing set X_0 may be identified with the set $\overline{X}_0$.

PROOF. *Necessity.* Assume K isomorphic to $B(\overline{X}_0)$. It is then evident that the minimal projections e_{x_0}, $x_0 \in X_0$, of K correspond in a 1-1, onto manner to the characteristic functions in $B(\overline{X}_0)$ of single points $\bar{x}_0$ of $\overline{X}_0$, that the indexing set X_0 may be identified with the set $\overline{X}_0$, and that the two conditions given above are satisfied with the e_A of condition (2) being the inverse images in K of the characteristic functions in $B(\overline{X}_0)$ of various subsets $\overline{A}_0$ of $\overline{X}_0$.

Sufficiency. Assume that the collection (now nonempty) e_{x_0}, $x_0 \in X_0$, of minimal projections in K satisfies the stated conditions. By the Gelfand-Neumark theory [1; 2] K is isomorphic (in a norm and * preserving manner) to an algebra $C(\overline{X})$ of all continuous complex-valued functions on a compact Hausdorff space $\overline{X}$. Let $\overline{X}_0$ denote the set of all points $\bar{x}_0$ which are open-closed sets in $\overline{X}$. Using again the theorem that for each point $\bar{y}$ in an open set O of $\overline{X}$ there exists an element c of $C(\overline{X})$ with $c(\bar{y}) = 1$ and $c(\bar{x}) = 0$ for $\bar{x} \notin O$, it is clear that the minimal projections e_{x_0}, $x_0 \in X_0$, of K correspond in a 1-1 onto manner to the collection $e_{\bar{x}_0}$, $\bar{x}_0 \in \overline{X}_0$, of all characteristic functions on single point, open-closed sets of $\overline{X}$, that X_0 and $\overline{X}_0$ may be identified, that (by condition 1) the points of $\overline{X}_0$ are dense in $\overline{X}$, and that (by condition (2)) each nonempty subset of points in $\overline{X}_0$ is contained in an open-closed subset of $\overline{X}$ which contains all and only those points of $\overline{X}_0$ that are in the given subset. It follows then by Theorem 1 that $C(\overline{X})$ and hence K is isomorphic to a $B(X_0^*)$ wherein the set X_0^* may be identified with $\overline{X}_0$ and thus with X_0.

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A NOTE ON UNSTABLE HOMEOMORPHISMS[1]

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In [1] W. R. Utz introduced the concept of an *unstable*[2] *homeomorphism* and raised the question of whether there exists an unstable homeomorphism of a compact continuum onto itself. In this note an example of such an homeomorphism will be given.

Let C denote the complex unit circle and for each $z \in C$, let $g(z) = z^2$. Then $g: C$ onto C determines an inverse limit space $\Sigma_2 = \{(a_0, a_1, a_2, \cdots) \mid$ for each non-negative integer i, $a_i \in C$ and $g(a_{i+1}) = a_i\}$. For $a, b \in \Sigma_2$, the function $\rho(a, b) = \sum_{i=0}^{\infty} |a_i - b_i|/2^i$ is a metric for Σ_2; Σ_2 is familiar as the "two-solenoid," and is a compact, indecomposable continuum. Define $f: \Sigma_2$ onto Σ_2 as follows: for each $a = (a_0, a_1, \cdots) \in \Sigma_2$, let $f(a) = [g(a_0), g(a_1), \cdots]$. Then $f(a) = (a_0^2, a_1^2, \cdots) = (a_0^2, a_0, a_1, \cdots)$, $f^{-1}(a) = (a_1, a_2, a_3, \cdots)$, and f is a homeomorphism of Σ_2 onto Σ_2.

To show that f is unstable, suppose that $a = (a_0, a_1, \cdots)$ and $b = (b_0, b_1, \cdots)$ are distinct points of Σ_2. Consider, as Case 1, that $a_0 \neq b_0$. Let $e^{i\theta} = a_0$, $e^{i\phi} = b_0$, where $0 \leqq \theta, \phi < 2\pi$. Then there exists a non-negative integer n such that the angle between the terminal rays of $2^n\theta$ and $2^n\phi$ is greater than $\pi/2$. Then $\rho[f^n(a), f^n(b)] \geqq |a_0^{2^n} - b_0^{2^n}| = |e^{i2^n\theta} - e^{i2^n\phi}| > 1$.

Case 2: for some integer $n > 0$, $a_n \neq b_n$, but $a_i = b_i$, for $0 \leqq i < n$. Then $f^{-n}(a) = (a_n, a_{n+1}, a_{n+2}, \cdots)$, $f^{-n}(b) = (b_n, b_{n+1}, b_{n+2}, \cdots)$, and there-

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[2] A homeomorphism f of a compact metric space X onto X is said to be *unstable* provided there exists a fixed positive number δ, such that if x and y are distinct points of X, then there exists an integer n, such that $\rho[f^n(x), f^n(y)]$ is greater than δ.