THE TITCHMARSH SEMI-GROUP

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Introduction. Let S be the set of all complex-valued functions defined on $\Omega=\{0, \pm 1, \pm 2, \cdots\}$. We suppose $p>1$, and consider the space S_p of all members a of S such that $\|a\|_p=(\sum_{\nu=-\infty}^{\infty} |a_\nu|^p)^{1/p}<\infty$. The letters α and λ will henceforth denote complex numbers.

The transformation T_α is defined as follows: if $a\in S_p$, then $T_\alpha a$ is the member c of S satisfying

$$[T_\alpha a]_n = c_n = \sum_{\nu=-\infty}^{\infty} (-1)^{n+\nu} \frac{\sin \alpha\pi}{(n+\alpha-\nu)\pi} a_\nu \qquad (n \in \Omega). \tag{1}$$

For any a in S_p, we define Ga to be the member x of S such that

$$[Ga]_n = x_n = \sum_{\nu=-\infty}^{\infty} (-1)^{n+\nu} \frac{1}{n-\nu} a_\nu, \qquad \nu \neq n \ (n \in \Omega). \tag{2}$$

M. Riesz [**10**] has shown that both T_α and G are bounded operators. An operator essentially identical to T_α was studied by E. C. Titchmarsh [**12**; **13**] in the case $\alpha=1/2$.

In this paper we prove that:

(i) T_α is an entire function.

(ii) $T_0 a=a$ and $T_\alpha T_\lambda a=T_{\alpha+\lambda}a$ for any a in S_p.

(iii) The operator G is the infinitesimal generator of what we call the *Titchmarsh semi-group* $\{T_\alpha | \alpha\}$. Moreover $T_\alpha=\exp \alpha G$ for all complex α.

These results are derived from the following property (A_p).

(A_p) *There is a nondecreasing function f such that* $\|T_\alpha\|_p \leqq f(|\operatorname{Im} \alpha|)$.

If $\operatorname{Im} \alpha=0$, it is readily inferred from (ii) and (A_p) that T_α is weakly-almost-periodic in the sense of Lorch [**6**]; its spectrum $\sigma(T_\alpha)$ is therefore on the circumference of the unit-circle.[1]

Our basic lemmas depend heavily upon the methods developed in [**12**]. W. Ferrar [**1**] has verified a special case of (ii). The operator T_α forms the basis of an article by H. Hadwiger [**2**]; in E. Hille's review [**4**] of [**2**] is found the remark that (i) and (ii) hold when $p=2$. A corresponding generalization of [**2**] is made possible by our removal of this restriction. Among various applications of T_α we mention: inter-

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[1] This follows from [**6**, p. 36]. We intend to show in a later article that $\sigma(iG) = [-\pi, \pi]$ and use these properties to illustrate a general theorem based on [**6**].

polation theory [1], fractional differentiation [2], and the derivation by Titchmarsh of the fundamental properties of the Hilbert transformation [12]. An extensive literature [3] deals with the operators obtained from (1) and (2) by imposing the conditions $n>0$, $\nu>0$. Some of the connections of this subject with problems in other fields are mentioned in [8; 7].

1. **Preliminaries.** As usual, $\|T_\alpha\|_p=\sup\|T_\alpha a\|_p$; $\|a\|_p=1$, $a\in S$. We shall consistently write $p'=p/(p-1)$. Let t^α be the member of S such that t_n^α is the nth Fourier coefficient of $\exp(-i\theta\alpha)$. Thus

$$t_n^\alpha=(-1)^n\xi_\alpha/(n+\alpha) \quad \text{if} \quad \xi_\alpha=(\sin\alpha\pi)/\pi \quad \text{and} \quad n\neq-\alpha,\ n\in\Omega;$$

moreover $t_{-\alpha}^\alpha=1$ if $\alpha\in\Omega$. In case $a\in S_p$ and $b\in S_{p'}$, then $b*a$ denotes the member x of S such that

$$[b*a]_n=x_n=\sum_{\nu=-\infty}^{\infty}b_\nu a_{n-\nu} \qquad \text{when } n\in\Omega.$$

Consequently $T_\alpha a=t^\alpha*a$. We shall show in §5 that $Ga=d*a$, where d_n is the derivative of t_n^α at $\alpha=0$. If $m\in\Omega$ then $[T_m a]_n=a_{m+n}$, so that

$$(3) \qquad T_0a=a \quad \text{and} \quad \|T_m\|_p=1 \qquad \text{when } m\in\Omega.$$

The Parseval and Fischer-Riesz theorems yield immediately (see [10; 14])

$$(4) \qquad \|T_\alpha\|_2\leqq\zeta(\alpha)=\exp(\pi|\operatorname{Im}\alpha|) \quad \text{and} \quad \|t^\alpha\|_2\leqq\zeta(\alpha).$$

1.1 Notation. From now on, we denote by $\mathcal{P}_p$ the set of all members c of S such that $c_n=0$ for all $|n|$ sufficiently large, and satisfying $\|c\|_p=1$.

We say that $\phi(\alpha)\in\mathcal{J}$ if $\phi(\alpha)\in[0,\infty)$ and if moreover ϕ is a function such that

(a) $\phi(\alpha)$ is bounded when $\alpha\in[0,1]$,

(b) there exists a nondecreasing function f mapping $[0,\infty)$ into itself, and satisfying $\phi(\alpha)\leqq f(|\operatorname{Im}\alpha|)$ whenever $\operatorname{Rl}\alpha=0$.

1.2 Remark.[2] If $\|x-y\|_p\leqq h_1(\alpha)\in\mathcal{J}$ and $\|x\|_p\leqq h_2(\alpha)\in\mathcal{J}$, then $\|y\|_p\leqq h_3(\alpha)\in\mathcal{J}$.

1.3 Lemma. *If* $\|T_\alpha\|_p<\infty$, *then* (ii) *holds.*

Proof. Suppose $a\in S_p$ and let K be an arbitrary pair (K_1, K_2) of members of Ω. We denote by $\{a; K\}$ the member x of S such that: $x_n=a_n$ if $K_1<n<K_2$, and $x_n=0$ otherwise $(n\in\Omega)$. Thus, if $n\in\Omega$,

[2] For convenience and brevity, the statement that there exists a function h such that $\|z\|_p\leqq h(\alpha)$ and $h(\alpha)\in\mathcal{J}$ will be expressed symbolically by writing $\|z\|_p\leqq h(\alpha)\in\mathcal{J}$.

$$(5)\qquad [T_\alpha a]_n = \lim\, [T_\alpha\{a; K\}]_n \qquad \text{as } K_1 \to -\infty,\ K_2 \to \infty.$$

By Hölder's inequality

$$\big|\, [T_\alpha c]_n \big| = \big|\, [t^\alpha * c]_n \big| \leqq \|t^\alpha\|_{p'} \|c\|_p.$$

Hence, taking $c = T_\lambda a - T_\lambda\{a;\, K\} = T_\lambda(a - \{a;\, K\})$,

$$(6)\quad \big|\, [T_\alpha(T_\lambda a)]_n - [T_\alpha(T_\lambda\{a; K\})]_n \big| \leqq \|t^\alpha\|_{p'} \|T_\lambda\|_p \|a - \{a; K\}\|_p.$$

It can easily be shown [10] that (ii) holds in the case $p=2$. But $\{a;\, K\} \in S_2$, and we can therefore replace, in (6), $T_\alpha T_\lambda\{a;\, K\}$ by $T_{\alpha+\lambda}\{a;\, K\}$. The conclusion is now obtained by taking $\lim K_1 \to -\infty$ and $\lim K_2 \to \infty$ of both sides of (6), using (5) and the fact that

$$\lim \|a - \{a; K\}\|_p = \lim \sum_{n=-\infty}^{K_1} |a_n|^p + \lim \sum_{n=K_2}^{\infty} |a_n|^p = 0.$$

2. **Basic lemmas.** In the present section, we prove that (A_p) holds for any p in $M = \{2, 2^2, 2^3, \cdots\}$. We henceforth write $s' = s/(s-1)$ and call d the member of S such that $d_0 = 0$, $d_n = (-1)^n/n$ when $n \neq 0$. If $a \in S$ and $b \in S$, it will be convenient to define ab and a^2 by $[ab]_n = a_n b_n$ and $[a^2]_n = a_n^2$ respectively. Note that if $s > 1$

$$(7)\qquad \|ab\|_s \leqq \|a\|_{2s} \|b\|_{2s} \quad \text{and} \quad \|a^2\|_s = \|a\|_{2s}^2.$$

2.1 Lemma. *Suppose $s \geqq 2$ and $\tau = 1, 2$. Then*

$$\|\{(t^\alpha)^\tau d\} * x\|_s \leqq \Psi_s^{(\tau)}(\alpha) \in \mathcal{J} \qquad \textit{when } x \in \mathcal{P}_s.$$

Proof. If $z = \{(t^\alpha)^\tau d\} * x$ and $n \in \Omega$, then by Hölder's inequality

$$|z_n|^s \leqq \|d\|_{s'}^s \sum_\nu |t_\nu^\alpha|^{s\tau} |x_{n-\nu}|^s = \|d\|_{s'}^s \sum_\theta |x_\theta|^s |t_{n-\theta}^\alpha|^{s\tau}.$$

Now $t_{n-\theta}^\alpha = t_n^{\alpha-\theta}$. We therefore[3] infer from (4) that

$$\|z\|_s \leqq \|d\|_{s'} \Big(\sum_\theta |x_\theta|^s \|t^{\alpha-\theta}\|_{s\tau}^{s\tau} \Big)^{1/s} \leqq \|d\|_{s'} \|x\|_s |\zeta(\alpha)|^\tau.$$

We conclude the proof by observing that $|\zeta(\alpha)|^\tau \in \mathcal{J}$.

2.2 Lemma. *Suppose $s \geqq 2$ and $\tau = 1, 2$. Then*

$$\|[\{(t^\alpha)^{\tau-1} \xi_\alpha d\} * x] - [(t^\alpha)^\tau * x]\|_s \leqq \Phi_s^{(\tau)}(\alpha) \in \mathcal{J} \qquad \textit{when } x \in \mathcal{P}_s.$$

Proof. Call $y = \{(t^\alpha)^{\tau-1} \xi_\alpha d\} - (t^\alpha)^\tau$. It is easily checked that

[3] Since $s\tau \geqq 2$, and thus $\|t^{\alpha-\theta}\|_{s\tau} \leqq \|t^{\alpha-\theta}\|_2 \leqq \zeta(\alpha)$.

$$y_n = (-1)^n \alpha (t_n^{\alpha})^{\tau} \cdot d_n - (t_0^{\alpha})^{\tau} \cdot t_n^0 \qquad (n \in \Omega),$$

so that if $x \in \mathcal{P}_s$,

$$\|y * x\|_s \leqq |\alpha| \cdot \|\{(t^{\alpha})^{\tau} d\} * x\|_s + |t_0^{\alpha}|^{\tau} \cdot \|t^0 * x\|_s.$$

But $t^0 * x = x$, $|t_0^{\alpha}|^{\tau} \in \mathcal{J}$, and by 2.1

$$\|y * x\|_s \leqq |\alpha| \Psi_s^{(\tau)}(\alpha) + |t_0^{\alpha}|^{\tau} = \Phi_s^{(\tau)}(\alpha) \in \mathcal{J}.$$

2.3 Lemma. *If $r \geqq 2$ there exists a function g_r with $g_r(\alpha) \in \mathcal{J}$ and such that for all a in $\mathcal{P}_{2r}$*

$$\|T_\alpha a\|_{2r}^2 - 2 \cdot \|T_\alpha\|_r (\|T_\alpha a\|_{2r} + g_r(\alpha)) \leqq f_r(\alpha) \in \mathcal{J}.$$

Proof. Suppose $a \in \mathcal{P}_{2r}$ and call $h = (T_\alpha a)^2 - [(t^\alpha)^2 * a^2]$. We first note that

$$h_n = \sum_{\nu} \sum_{\theta \neq \nu} t_\nu^{\alpha} t_\theta^{\alpha} a_{n-\nu} a_{n-\theta}. \tag{8}$$

By (7), $\|a^2\|_r = \|a\|_{2r}^2 = 1$, so that $a^2 \in \mathcal{P}_r$. We can therefore conclude from 2.2, 2.1, and 1.2 that $\|(t^\alpha)^2 * a^2\|_r \leqq f_r(\alpha) \in \mathcal{J}$. From the definition of h now follows that

$$\|T_\alpha a\|_{2r}^2 - \|h\|_r \leqq f_r(\alpha) \in \mathcal{J}. \tag{9}$$

Suppose $x \in S$; if we define $-x$ by $[-x]_n = (-1)^n x_n$, then $\|-x\|_r = \|x\|_r$ and $(-z) * x = -[z * (-x)]$. This enables us to derive from (7) that

$$\|(-t^\alpha) * (ac)\|_r = \|T_\alpha(-ac)\|_r \leqq \|T_\alpha\|_r \|ac\|_r \leqq \|T_\alpha\|_r \|c\|_{2r}. \tag{10}$$

It is immediately verified that $t_\nu^\alpha t_\theta^\alpha = \xi_\alpha([-t^\alpha]_\nu d_m + [-t^\alpha]_\theta d_{-m})$ if $m = \theta - \nu \neq 0$ so that, by (8),

$$h_n = 2 \sum_{\nu} [-t^\alpha]_\nu a_{n-\nu} \sum_m \xi_\alpha d_m a_{(n-\nu)-m} = 2[(-t^\alpha) * (a\{\xi_\alpha d * a\})]_n.$$

Using now (10) and 2.2,

$$\|h\|_r \leqq 2\|T_\alpha\|_r \|\xi_\alpha d * a\|_{2r} \leqq 2\|T_\alpha\|_r (\|t^\alpha * a\|_{2r} + \Phi_{2r}^{(1)}(\alpha)).$$

The conclusion follows from (9).

2.4 Final lemma. *Whenever $\|T_\alpha\|_r \leqq \phi_r(\alpha) \in \mathcal{J}$ for some $r \geqq 2$, then $\|T_\alpha\|_{2r} \leqq \phi_{2r}(\alpha) \in \mathcal{J}$.*

Proof. If $a \in \mathcal{P}_{2r}$ and $x = T_\alpha a$, then by 2.3

$$\|x\|_{2r}^2 - 2\phi_r(\alpha)\|x\|_{2r} \leqq 2\phi_r(\alpha) \cdot g_r(\alpha) + f_r(\alpha) = h_1(\alpha) \in \mathcal{J}.$$

Adding $(\phi_r(\alpha))^2$ to both sides of the inequality,

$$\{\|x\|_{2r}^2 - \phi_r(\alpha)\}^2 \leqq (\phi_r(\alpha))^2 + h_1(\alpha) = h_2(\alpha) \in \mathcal{J}.$$

This yields the conclusion

$$\|T_\alpha a\|_{2r} = \|x\|_{2r} \leqq \phi_r(\alpha) + (h_2(\alpha))^{1/2} = \phi_{2r}(\alpha) \in \mathcal{J}.$$

2.5 THEOREM. (A_s) *holds for any s in M.*

PROOF. From (4) and 2.4 we can conclude that for any s in M, $\|T_\lambda\|_s \leqq \phi_s(\lambda) \in \mathcal{J}$. Set $\alpha = \alpha^0 + \alpha' + i\alpha''$, where $\alpha^0 \in \Omega$, $\alpha' \in [0, 1]$, and $\alpha'' \in (-\infty, \infty)$; then by 1.3 and (3),

$$\|T_\alpha\|_s = \|T_{\alpha^0} T_{\alpha'+i\alpha''}\|_s \leqq \|T_{\alpha'+i\alpha''}\|_s \leqq \phi_s(\alpha')\|T_{i\alpha''}\|_s.$$

Now $\phi_s(\lambda) \in \mathcal{J}$, $\alpha' \in [0, 1]$, and $\phi_s(\alpha')$ is therefore[4] bounded by some number k. Moreover, $\|T_{i\alpha''}\|_s \leqq \phi_s(i\alpha'')$ and we infer from 1.1(b) and Rl $i\alpha'' = 0$ that there exists a nondecreasing function f such that $\phi_s(i\alpha'') \leqq f(|\alpha''|)$. Collecting results: $\|T_\alpha\|_s \leqq k \cdot f(|\alpha''|)$.

3. The main theorems.

3.1 LEMMA. *If $p > 1$, there exists a member s of M such that $\|T_\alpha\|_p \leqq \|T_\alpha\|_s$.*

We have already indicated that T_α is essentially the operator studied by Titchmarsh [12] in the case $\alpha = 1/2$; 3.1 is the generalization to complex α of the statement (2.47) found in [12, p. 332]. We omit the verification of the fact that every step in Titchmarsh's proof of (2.47) can be directly extended, and quote his assertion [12, p. 323] that "the theory (of T_α) presents no features which do not occur in the case $\alpha = 1/2$."

THEOREM I. *For any $p > 1$ there exists a nondecreasing function f such that $\|T_\alpha\|_p \leqq f(|\mathrm{Im}\ \alpha|)$ for any complex α.*

PROOF. From 3.1 we have $\|T_\alpha\|_p \leqq \|T_\alpha\|_s$ for some s in M. But by 2.5, there exists a nondecreasing f such that $\|T_\alpha\|_s \leqq f(|\mathrm{Im}\ \alpha|)$. The conclusion follows.

THEOREM II. *If $p > 1$, then $T_\alpha(T_\lambda a) = T_{\alpha+\lambda} a$ for any a in S_p.*

This is an obvious consequence of 1.3 and Theorem I.

4. **Analyticity of T_α.** Let us denote by $\mathfrak{E}_p$ the set of all bounded linear transformations of S_p into itself. We say with Hille [5, p. 53] that a member V_α of $\mathfrak{E}_p$ is an *entire function* if $\phi(V_\alpha x)$ is an entire

[4] By 1.1(a).

function of α for every choice of x in S_p, and for every ϕ in the adjoint space S_p^*.

In such a case, there exists [5, p. 53] a member V'_α of $\mathfrak{E}_p$ such that

$$V'_\alpha x = \lim_{\epsilon\to 0} \left\{ V_{\alpha+\epsilon}x - V_\alpha x \right\} \frac{1}{\epsilon} \qquad \text{for all } x \text{ in } S_p. \tag{11}$$

Moreover

$$[V'_\alpha x]_n = \frac{d}{d\alpha}[V_\alpha x]_n \qquad \text{for every } n \text{ in } \Omega. \tag{12}$$

We derive (12) from (11) by observing that for any ϕ in S_p^*

$$\phi(V'_\alpha x) = \lim_{\epsilon\to 0} \left\{ \phi(V_{\alpha+\epsilon}x) - \phi(V_\alpha x) \right\} \frac{1}{\epsilon} = \frac{d}{d\alpha}\phi(V_\alpha x).$$

Suppose $n \in \Omega$; the above holds in particular for the member ϕ_n of S_p^* defined by $\phi_n(c) = c_n$ $(c \in S_p)$.

THEOREM III. *The operator T_α is an entire function, and for any a in S_p we have*

$$[T'_\alpha a]_n = \sum_{\nu=-\infty}^{\infty} \left(\frac{d}{d\alpha} t_\nu^\alpha \right) a_{n-\nu} \qquad \textit{when } n \in \Omega. \tag{13}$$

PROOF. It is easily seen [1, p. 231] that the series representing $[T_\alpha a]_n$ is uniformly convergent in any bounded region. Hence $[T_\alpha a]_n$ is an entire function of α, and the series in (13) represents therefore the derivative of $[T_\alpha a]_n$.

From (12) now follows that the theorem will be proved once we have established that T_α is an entire function. To that effect, we note that any closed and bounded region $\mathcal{D}$ can be included in a suitable square $|\mathrm{Im}\ \alpha| < \tau$, $|\mathrm{Rl}\ \alpha| < \tau$ and therefore, by Theorem I

$$\|T_\alpha\|_p \leqq f(|\mathrm{Im}\ \alpha|) \leqq f(\tau) \qquad \text{for all } \alpha \text{ in } \mathcal{D}. \tag{14}$$

From the analyticity of $[T_\alpha a]_n = \psi_\alpha^{(n)}(a)$ follows that the members $\psi_\alpha^{(n)}$ of S_p^* are analytic. We now refer to [11] for the fact that this allows us to deduce from (14) the analyticity of T_α in the arbitrary region $\mathcal{D}$. This completes the proof.

5. **The generator.** Since $\mathfrak{E}_p$ forms a Banach space with norm $\|\cdot\|_p$, we can define $\exp \alpha G$ for any G in $\mathfrak{E}_p$. The continuity of T_α is readily inferred [5, p. 53] from Theorem III. Since further $T_0 a = a$ and $T_\alpha T_\lambda = T_{\alpha+\lambda}$, we can conclude from [9] that, when α is real

$$(15) \qquad T_\alpha = \exp \alpha G, \qquad \text{where } G = T_0'.$$

Both T_α and $\exp \alpha G$ being entire functions, we can extend the validity of (15) to all complex values of α by analytic continuation [5, p. 58]. From (15) we see that $G \in \mathfrak{C}_p$; G is the infinitesimal generator of the analytical group $\{T_\alpha | \alpha\}$.

The definition of t^α readily yields[5]

$$\left[\frac{d}{d\alpha} t_n^\alpha\right]_{\alpha=0} = d_n \qquad (n \in \Omega).$$

It now follows from (15) and (13) that

$$[Ga]_n = \sum_{\nu=-\infty}^{\infty} d_\nu a_{n-\nu} = [d * a]_n \qquad (a \in S_p,\ n \in \Omega).$$

Hence G is identical to the operator defined by (2); this fulfills the aims set forth in the introduction.

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[5] See §2 for the definition of d.