INTEGRABLE POTENTIALS AND HALF-LINE SPECTRA[1]

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1. In the differential equation

$$x'' + (\lambda - f)x = 0, \tag{1}$$

let $f = f(t)$ be a real-valued, continuous function on $0 \leqq t < \infty$ and suppose that λ is a real parameter. If (1) is of the limit-point type, then (1) and a boundary condition of the type

$$x(0)\cos\alpha + x'(0)\sin\alpha = 0, \qquad 0 \leqq \alpha < \pi, \tag{2}$$

determine, for every fixed α, a boundary value problem on $0 \leqq t < \infty$ with a spectrum (of λ-values) $S = S_\alpha$ [7]. It is known that the set S' consisting of the set of cluster points of S_α is independent of α; loc. cit. p. 251. The following theorem will be proved:

(*) *If $f(t)$ denotes a real-valued, continuous function on the half-line $0 \leqq t < \infty$ satisfying the condition*

$$\int_0^\infty f(t)dt \text{ converges} \qquad \left(\int_0^\infty = \lim_{T\to\infty} \int_0^T\right), \tag{3}$$

then (1) *is of the limit-point type and*

$$S' = [0, \infty). \tag{4}$$

It is noteworthy that (3) may exist only conditionally and that

$$\int_0^\infty |f(t)|\, dt < \infty \tag{3'}$$

is not assumed. Actually, if (3′) holds, much more is known. In fact, in this case, there exist asymptotic formulas for the solutions of (1) when $\lambda > 0$ ([8, p. 421]; cf. also [7, p. 258], in case $f(t)\to 0$ as $t\to\infty$) which guarantee, in particular, that $0 \leqq \lambda < \infty$ is in the continuous spectrum for every boundary value problem determined by (1) and (2). Obviously, the requirement (3) is compatible with $T^{-1}\int_0^T |f(t)|\,dt \to \infty$, as $T\to\infty$, and, in fact, even with the requirement that $\int_0^T |f(t)|\,dt \to \infty$ arbitrarily fast. Thus, if $\phi(t)$ denotes any positive function satisfying $\phi(t)\to\infty$ as $t\to\infty$, there exists a continuous function $f(t)$ on $0 \leqq t < \infty$ satisfying (3) and $\phi(T) = o(\int |f_0^T(t)|\,dt)$, as $T\to\infty$.

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On the other hand, most of the criteria for (4) or $S' \supset [0, \infty)$ involve $|f(t)|$ rather than $f(t)$, and, as a consequence, require that $f(t)$ be close to zero "on the average." For instance, it is known that

$$(5) \qquad T^{-1}\int_0^T |f(t)|\,dt \to 0, \qquad T\to\infty,$$

is enough to guarantee that $S' \supset [0, \infty)$, although

$$\limsup T^{-1}\int_0^T |f(t)|\,dt < \infty$$

is not; cf. [**3**, p. 80]. Moreover, (5) is compatible with $S' = (-\infty, \infty)$; cf. [**3**].

2. **Proof of** (*). Since f satisfies (3), it is clear that $\int_0^T(\lambda - f(t))dt \to \infty$ as $T\to\infty$ whenever $\lambda > 0$. It follows that (1) is oscillatory (i.e., every nontrivial solution possesses an infinity of zeros clustering at $+\infty$) whenever $\lambda > 0$; [**10**], cf. also [**4**]. Next, it will be shown that, in view of (3), the equation (1) is nonoscillatory whenever $\lambda < 0$. (It is of interest to note here that there are known necessary and sufficient conditions in order that an equation (1) be oscillatory; cf., e.g., [**5**; **9**]. In the present case it will be convenient for later use to give the direct argument below.)

Suppose first that λ is arbitrary and that (1) possesses an oscillatory solution $x = x(t) (\not\equiv 0)$ with zeros tending to infinity. If $S < T$ denote two zeros of $x(t)$, a multiplication of (1) by x followed by an integration leads to

$$(6) \qquad \int_S^T x'^2 dt = \lambda \int_S^T x^2 dt - \int_S^T f x^2 dt.$$

An integration by parts of the second integral on the right side of the equation (6) yields

$$(7) \qquad \int_S^T f x^2 dt = -2\int_S^T x x' F(t) dt, \qquad F(t) = \int_0^t f(s) ds.$$

In view of (3), $F(t) = \text{const.} + o(1)$ as $t\to\infty$, and an application of the Schwarz inequality to the second integral of (7) now implies

$$\int_S^T x'^2 dt = \lambda \int_S^T x^2 dt + o\left(\int_S^T x^2 dt \int_S^T x'^2 dt\right)^{1/2},$$

and hence,

$$(8) \qquad A = \lambda + o(A^{1/2}), \quad \text{where} \quad A = \int_S^T x'^2 dt \Big/ \int_S^T x^2 dt,$$

where the "o term" refers to $S\to\infty$. It readily follows from (8) that $\lambda\geqq 0$ and so (1) must be nonoscillatory whenever $\lambda<0$.

It follows from the last result that (1) is of the limit-point type and that, in addition, $S'\subset[0, \infty)$; [1], cf. also [2]. There remains to be shown that the half-line $\lambda\geqq 0$ belongs to S'. To this end, consider any boundary condition (2) for a fixed value α and let

$$m_\alpha(\lambda) = \min |\lambda - \mu|,$$

when μ is in the (closed) set S_α. It will be shown that

$$m_\alpha(\lambda) \equiv 0 \qquad \text{for } \lambda > 0 \text{ (hence for } \lambda \geqq 0), \tag{9}$$

and so (4) will follow.

Let $g=g(t)$ denote any function of class C^2 on the finite interval $0\leqq t\leqq T$ and satisfying the boundary conditions (2) and

$$g(T) = g'(T) = 0. \tag{10}$$

Then the argument of [6, pp. 579–580] shows that

$$m_\alpha^2(\lambda)\int_0^T g^2dt \leqq \int_0^T (L(g)+\lambda g)^2dt \qquad (L(x)\equiv x''-fx). \tag{11}$$

Next, let μ and ϵ be positive and suppose that $g(t)=y(t)h(t)$, where $h(t)=\cos(\mu^{1/2}t)$ and $y(t)$ is a nontrivial (oscillatory) solution of (1) for $\lambda=\epsilon$, so that $L(y)+\epsilon y=0$, and satisfying (2) for $x=y$. Next, let T be chosen so that

$$y(T) = 0. \tag{12}$$

In addition, since (1) is of the limit-point type, the number ϵ can be chosen arbitrarily small and so that the function y satisfies

$$\int_0^\infty y^2dt = \infty; \tag{13}$$

cf. [7]. It will be supposed that $\mu=\mu(T)$ is chosen so that

$$\cos(\mu^{1/2}T) = 0; \tag{14}$$

hence, as a consequence of (12) and the relation $g'=y'h+yh'$, $g(t)$ also satisfies (10). In view of

$$L(g)+\lambda g = (\lambda-\mu-\epsilon)hy + 2y'h', \tag{15}$$

the relation (11) and the inequality $(a+b)^2\leqq 2(a^2+b^2)$ now yield

$$m_\alpha^2(\lambda)\int_0^T h^2y^2dt \leqq \text{const.} \int_0^T [\mu y'^2 + (\lambda-\mu-\epsilon)^2y^2]dt. \tag{16}$$

Next, let $T=T_1<T_2<\cdots$ denote the positive zeros of $y=y(t)$

and choose $\mu_n=\mu(T_n)$ (hence $h=h_n$) so that (14) holds for $T=T_n$ and $\mu_n\to\lambda(>0)$. (That this can be done is clear.) It follows from (16) that, as $n\to\infty$,

$$(17)\qquad m_\alpha^2(\lambda)\leqq \text{const.}\limsup\left[\int_0^{T_n}(\epsilon^2y^2+\lambda y'^2)dt\Big/\int_0^{T_n}h_n^2y^2dt\right].$$

A calculation like that of [6, p. 581], together with (12), yields

$$(18)\qquad \int_0^{T_n}h_n^2y^2\,dt\geqq\frac{1}{2}\int_0^{T_n}y^2dt-\frac{1}{2}\mu_n^{-1/2}\left(\int_0^{T_n}y'^2dt\int_0^{T_n}y^2dt\right)^{1/2}.$$

If use is made of (13), a calculation similar to that used in obtaining (8) shows that $A=\epsilon+o(A^{1/2})$, as $T_n\to\infty$, where

$$A=A_n=\int_0^{T_n}y'^2dt\Big/\int_0^{T_n}y^2dt.$$

This implies however that $A(T_n)<\text{const. }\epsilon$ for T_n large, and hence, by (18), $\int_0^{T_n}h_n^2y^2dt\geqq\text{const.}\int_0^{T_n}y^2dt>0$ for T_n large and for a sufficiently small ϵ. Finally, relation (17) now implies $m_\alpha^2(\lambda)\leqq\text{const.}(\epsilon^2+\epsilon\lambda)$. Since $\epsilon>0$ can be chosen arbitrarily small, relation (9) follows and the proof of (*) is complete.

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