BEHAVIOR OF SOLUTIONS OF SECOND ORDER SELF-ADJOINT DIFFERENTIAL EQUATIONS

J. H. BARRETT

Introduction. In this paper the modified polar coordinate transformation,

$$\text{(I)} \qquad y(x) = \rho(x) \sin \theta(x), \qquad y'(x) = \frac{w(x)}{r(x)} \rho(x) \cos \theta(x),$$

will be applied to the self-adjoint equation,

$$(1) \qquad (ry')' + qy = 0.$$

For $x \geqq a$, let $q(x)$ be a function of class C, $r(x)$ be a positive function of class C, $w(x)$ be a positive function of class C', and $y(x)$ be a nontrivial solution of equation (1).[1] The reader can show that there exist functions $\rho(x)$ and $\theta(x)$ of class C' which satisfy (I) and $\rho(x) > 0$. Furthermore,

$$(\text{II}_1) \qquad \rho' = \rho\left[\left(\frac{w}{r} - \frac{q}{w}\right)\frac{\sin 2\theta}{2} - \frac{w'}{w}\cos^2\theta\right]$$

and

$$(\text{II}_2) \qquad \theta' = \frac{1}{2}\left(\frac{w}{r} + \frac{q}{w}\right) + \frac{1}{2}\left(\frac{w}{r} - \frac{q}{w}\right)\cos 2\theta + \frac{w'}{2w}\sin 2\theta.$$

The transformation (I) is an extension of the polar transformation

$$(\text{I}') \qquad y(x) = \rho(x) \sin \theta(x), \qquad y'(x) = \rho(x) \cos \theta(x)$$

of the normal form of the ordinary wave differential equation

$$(1') \qquad y'' + q(x)y = 0$$

which was introduced by Prüfer [6]. For well-known applications of (I′) to the self-adjoint equation (1) the reader is referred to [2, pp. 161–167] and [3, pp. 274–281]. W. M. Whyburn [8] has used this transformation in studying solutions of a system of two first order nonlinear equations. More recently F. V. Atkinson [1] has employed

Presented to the Society, April 24, 1954 under the title *Oscillation and boundedness of solutions of* $y''+q(x)=0$; received by the editors May 25, 1954 and, in revised form, August 10, 1954.

[1] That is, on $x \geqq a$, $y(x)$ satisfies (1), y and ry' are of class C', and y is not identically zero.

a special case of (I) in transforming (1′), namely,

$$\text{(I}''\text{)} \qquad y(x) = \rho(x)\cos\theta(x), \qquad \frac{y'(x)}{(q(x))^{1/2}} = \rho(x)\sin\theta(x).$$

The first section consists of consequences of the first equation (II_1), the ρ-equation. Certain theorems on functional bounds for solutions of equations (1) are given which are extensions of results of Levinson [5] and Leighton [4]. In the second section the equation (II_2), the θ-equation, will be used to establish sufficient conditions for oscillation of solutions of (1) and these are compared with necessary conditions established by Leighton [4].

Finally, an asymptotic form of solutions of (1) is obtained.

1. **Boundedness.** The first theorem is an extension of a theorem of Levinson.

THEOREM 1. *If* $w'(x) \leqq 0$, $Q(x) = \int_a^x |w/r - q/w|\,dt$, $x \geqq a$, *then*

$$\text{(III)} \qquad \rho(a)\exp\left[-Q(x)/2\right] \leqq \rho(x) \leqq \frac{\rho(a)w(a)}{w(x)}\exp\left[Q(x)/2\right],$$

and for each solution $y(x)$ *of* (1):

$$(2) \qquad |y(x)| \leqq \frac{\rho(a)w(a)}{w(x)}\exp\left[Q(x)/2\right],$$

$$(3) \qquad |y'(x)| \leqq \frac{\rho(a)w(a)}{r(x)}\exp\left[Q(x)/2\right].$$

PROOF. Equation (II_1) yields the inequalities:

$$-\frac{1}{2}\left|\frac{w}{r} - \frac{q}{w}\right| \leqq \frac{\rho'}{\rho} \leqq \frac{1}{2}\left|\frac{w}{r} - \frac{q}{w}\right| - \frac{w'}{w}$$

from which (III) is obtained by integration. Inequalities (2) and (3) follow from (III).

COROLLARY 1.1. *If, in addition to the hypotheses of Theorem* 1, $Q(x)$ *is bounded for* $x \geqq a$, *then for each solution* $y(x)$ *of* (1):

$$(4) \qquad y(x) = O\left(\frac{1}{w(x)}\right) \quad \textit{and} \quad y'(x) = O\left(\frac{1}{r(x)}\right), \qquad \textit{as } x \to \infty.$$

Let $w = (qr)^{1/2}$, then $Q(x) = 0$ and a result of Leighton is established:

COROLLARY 1.2. *If* $q(x) > 0$, $x \geqq a$, $q(x)\cdot r(x)$ *is of class* C' *and* $(qr)' \leqq 0$, *then*

$$(5') \qquad y(x) = O\left(\frac{1}{(rq)^{1/2}}\right) \quad \textit{and} \quad y'(x) = O\left(\frac{1}{r}\right) \qquad \textit{as } x \to \infty.$$

COROLLARY 1.3. *If, in addition to the hypotheses of Corollary* 1.2, $w(x)$ *(or* $r(x)$*) is bounded away from zero, then* $y(x)$ *(or* $y'(x)$*) is bounded.*

2. **Oscillation.** Note that $y(x)$ has a zero only when $\theta(x)$ is a multiple of π. If $\theta(x)$ is equal to an integral multiple of π at $x=x_1$, then $\theta'(x_1)=w(x_1)/r(x_1)>0$ and, therefore, if $\theta(x)$ takes on integral multiples of π for infinitely many values of x, then $\lim_{x\to\infty}\theta(x)=\infty$. Therefore, in order that $y(x)$ be oscillatory it is necessary and sufficient that

$$\lim_{x\to\infty}\theta(x) = \infty.$$

The θ-equation (II_2) yields the inequality:

$$(\mathrm{IV}) \qquad \theta' \geqq \frac{1}{2}\left(\frac{w}{r}+\frac{q}{w}\right) - \frac{1}{2}\left|\frac{w}{r}-\frac{q}{w}\right| - \frac{1}{2}\,\frac{|w'|}{w}$$

from which the following theorem is readily obtained.

THEOREM 2. *If there exists a positive function* $w(x)$ *of class* C' *on* $x \geqq a$ *such that*

$$\lim_{x\to\infty}\int_a^x\left[\left(\frac{w}{r}+\frac{q}{w}\right) - \left|\frac{w}{r}-\frac{q}{w}\right| - \frac{|w'|}{w}\right]dt = \infty,$$

then every nontrivial solution of (1) *is oscillatory.*

As in Corollary 1.2, the special choice $w=(qr)^{1/2}$ gives a simplified form of (IV) from which the following *sufficient* condition for oscillation is derived:

COROLLARY 2.1. *If* $q(x)>0$, $(qr)'\leqq 0$ *on* $x\geqq a$, *and*

$$\lim_{x\to\infty}\left[\int_a^x\left(\frac{q}{r}\right)^{1/2}dt + \frac{1}{4}\ln(q(x)r(x))\right] = \infty$$

then every nontrivial solution of (1) *is oscillatory.*

It is of interest to compare this sufficient condition with Leighton's corresponding necessary condition

$$\lim_{x\to\infty}\int_a^x\left(\frac{q}{r}\right)^{1/2}dt = \infty.$$

This corollary establishes that Leighton's condition is also sufficient for cases where the function qr is bounded away from zero. However, in many interesting cases,

$$\lim_{x\to\infty} r(x)q(x) = 0,$$

for example, the Euler equation where $q(x)=k/x^2$ and $r(x)=1$. For this example Corollary 2.1 shows that oscillation occurs for $k>1/4$.

3. Asymptotic behavior.

THEOREM 3. *If k is a positive number such that $\int_a^\infty |k/r-q/k|\,dt<\infty$, $\beta(x)=(1/2)\int_a^x(k/r+q/k)dt$, and $y(x)$ is any nontrivial solution of* (1), *then there exists a positive number A and a number α such that*

$$\lim_{x\to\infty} [y(x) - A \sin(\beta(x) + \alpha)] = 0.$$

PROOF. Let $w=k$ in equations (II). From a well-known theorem, it follows that $(1/2)\int_a^\infty(k/r-q/k)\sin 2\theta dt$ exists (i.e. is finite). Call this value A_1. Then since $\rho(x)$ satisfies (II_1):

$$\lim_{x\to\infty} \rho(x) = \rho(a)\cdot e^{A_1} = A > 0.$$

Furthermore, from (II_2):

$$\theta(x) = \theta(a) + \beta(x) + \frac{1}{2}\int_a^x (k/r - q/k)\cos 2\theta dt.$$

Hence, if $\alpha=\theta(a)+(1/2)\int_a^\infty(k/r-q/k)\cos 2\theta dt$, then

$$\lim_{x\to\infty} [\theta(x) - \beta(x) - \alpha] = 0.$$

Finally, it follows that

$$\lim_{x\to\infty} [y(x) - A \sin(\beta(x) + \alpha)] = 0.$$

Under the hypothesis of Theorem 3, the oscillation or nonoscillation of $y(x)$ depends on whether or not $\beta(x)\to\infty$ as $x\to\infty$. If

$$\lim_{x\to\infty} \int_a^x dt/r(t) = \infty,$$

then $\lim_{x\to\infty} \int_a^x q(t)dt=\infty$, since $\left|\int_a^x(k/r)dt-\int_a^x(q/k)dt\right| \leqq \int_a^x|k/r-q/k|\,dt$. Hence $\beta(x)\to\infty$ as $x\to\infty$, and there exists a sequence $\{x_n\}$

such that $\beta(x_n)+\alpha=(4n-3)\pi/2$, $x_n\to\infty$, and $\lim_{n\to\infty} y(x_n)=A>0$. This result is summarized in

COROLLARY 3.1. *If, in addition to the hypotheses of Theorem* 3,

$$\lim_{x\to\infty}\int_a^x \frac{dt}{r(t)} = \infty,$$

then $\limsup_{x\to\infty} y(x)>0$, *and* $y(x)$ *oscillates as* $x\to\infty$.

On the other hand, if $\int_a^\infty dt/r(t)<\infty$ and since $|q/k|\leqq k/r+|q/k-k/r|$, then $\int_a^\infty |q(t)|\,dt<\infty$ and $\lim_{x\to\infty}\beta(x)$ exists. Therefore, the next result follows easily.

COROLLARY 3.2. *If, in addition to the hypotheses of Theorem* 3,

$$\int_a^\infty \frac{dt}{r(t)} < \infty,$$

then $\lim_{x\to\infty} y(x)$ *and* $\lim_{x\to\infty} r(x)y'(x)$ *exist.*

For a different proof of an equivalent result by Wintner see [7, p. 58].

BIBLIOGRAPHY

1. F. V. Atkinson, *On second order linear oscillators*, Univ. Nac. Tucuman, Revista A vol. 8 (1951) pp. 71–87.

2. L. Bieberbach, *Theorie der Differentialgleichungen*, 3d ed., New York, Dover 1944.

3. E. Kamke, *Differentialgleichungen reeller Funktionen*, New York, Chelsea, 1947.

4. W. Leighton, *On self-adjoint differential equations of second-order*, J. London Math. Soc. vol. 27 (1952) pp. 37–47.

5. N. Levinson, *The growth of solutions of a differential equation*, Duke Math. J. vol. 8 (1941) pp. 1–10.

6. H. Prüfer, *Neue Herleitung der Sturm-Liouvilleschen Reihenentwicklung stetiger Funktionen*, Math. Ann. vol. 95 (1926) pp. 499–518.

7. A. Wintner, *Asymptotic integrations of the adiabatic oscillator in its hyperbolic range*, Duke Math. J. vol. 15 (1948) pp. 55–67.

8. W. M. Whyburn, *Existence and oscillation theorems for non-linear differential systems of the second order*, Trans. Amer. Math. Soc. vol. 30 (1929) pp. 848–854.

UNIVERSITY OF DELAWARE