A BASIC SET OF HOMOGENEOUS HARMONIC POLYNOMIALS IN k VARIABLES

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1. The authors present basic sets of:

(a) homogeneous harmonic polynomials of degree n in k variables, $k \geqq 3$;

(b) associated polynomial solutions of the wave equation, and

(c) analogous solutions for $\sum_{j=1}^{k} (\partial^s u)/(\partial x_j^s) = 0$, $s = 3, 4, \cdots$.

2. For any set of non-negative integers (b_j) such that $b_1 \leqq 1$ and $\sum_{j=1}^{k} b_j = n$, let

$$H^{n}_{b_1 b_2 \cdots b_k}(x_1, x_2, \cdots, x_k)$$

$$= \sum (-1)^{[a_1/2]} \frac{n!}{\prod_{j=1}^{k} a_j!} \cdot \frac{\left[\frac{a_1}{2}\right]!}{\prod_{j=2}^{k} \left(\frac{b_j - a_j}{2}\right)!} \prod_{j=1}^{k} x_j^{a_j} \tag{1}$$

where the summation is extended over all (a_j) such that:

(1) $a_j \equiv b_j \bmod 2$, $j = 1, 2, \cdots, k$,

(2) $\sum_{j=1}^{k} a_j = n$,

(3) $a_j \leqq b_j$, $j = 2, 3, \cdots, k$.

The polynomials (1) form a basic set of homogeneous harmonic polynomials in k variables. The proof is given in three parts.

A. The polynomials (1) are linearly independent since each contains exactly one different nonvanishing term of the monomials $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$, $\sum_{j=1}^{k} a_j = n$, $a_1 \leqq 1$.

Moreover, since the number of terms in $(\sum_{j=1}^{k} x_j)^n$ is

$$\binom{n + k - 1}{k - 1},$$

the total number of monomials of type $x_2^{a_2} x_3^{a_3} \cdots x_k^{a_k}$, $\sum_{j=2}^{k} a_j = n$, and of type $x_1 x_2^{a_2} x_3^{a_3} \cdots x_k^{a_k}$, $\sum_{j=2}^{k} a_j = n - 1$, is

$$\binom{n + k - 2}{k - 2} + \binom{n + k - 3}{k - 2} = \binom{n + k - 3}{k - 3}\left(\frac{2n}{k - 2} + 1\right).$$

Thus the polynomials (1) are

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$$\binom{n+k-3}{k-3}\left(\frac{2n}{k-2}+1\right)$$

in number.

B. They are harmonic. Let c_j, $j=1, \cdots, k$, be such that

(1) $c_j \equiv b \bmod 2$,

(2) $c_j \leqq b_j$, $j=2, 3, \cdots, k$, and

(3) $\sum_{j=1}^k c_j = n-2$.

The coefficient $B_{c_1, c_2, \cdots, c_k}$ of $\prod_{j=1}^k x_j^{c_j}$ in $\nabla^2 H_{b_1, \cdots, b_k}^n (x_1, \cdots, x_k)$ is given by

$$B_{c_1, \cdots, c_k} = (-1)^{[c_1/2]+1} \frac{n!}{\prod_{j=1}^k c_j!} \cdot \left[\frac{\left[\frac{c_1}{2}+1\right]!}{\prod_{j=1}^k \left(\frac{b_j - c_j}{2}\right)!} - \frac{\sum_{j=2}^k \left[\frac{c_1}{2}\right]! \left(\frac{b_j - c_j}{2}\right)}{\prod_{j=2}^k \left(\frac{b_j - c_j}{2}\right)!} \right]$$

$$= (-1)^{[c_1/2]+1} \frac{n! \left[\frac{c_1}{2}\right]!}{\prod_{j=1}^k c_j! \prod_{j=2}^k \left(\frac{b_j - c_j}{2}\right)!} \cdot \left[\left[\frac{c_1}{2}\right] + 1 - \frac{1}{2}\sum_{j=2}^k b_j + \frac{1}{2}\sum_{j=2}^k c_j\right]$$

$$= (-1)^{[c_1/2]+1} \frac{n! \left[\frac{c_1}{2}\right]!}{\prod_{j=1}^k c_j! \prod_{j=2}^k \left(\frac{b_j - c_j}{2}\right)!} \cdot \left[\left[\frac{c_1}{2}\right] + 1 - \frac{1}{2}(n - b_1) + \frac{1}{2}(n - 2 - c_1)\right]$$

$$= (-1)^{[c_1/2]+1} \frac{n! \left[\frac{c_1}{2}\right]!}{\prod_{j=1}^k c_j! \prod_{j=2}^k \left(\frac{b_j - c_j}{2}\right)!} \cdot \left[\left[\frac{c_1}{2}\right] - \frac{1}{2}(c_1 - b_1)\right] \equiv 0.$$

C. For a general homogeneous polynomial H_k^n of degree n in k variables the vanishing of the Laplacian $\nabla^2 H_k^n$ provides

$$\binom{n+k-3}{k-1} \text{ equations on the } \binom{n+k-1}{k-1} \text{ coefficients}$$

of H_k^n. Thus the number of linearly independent homogeneous harmonic polynomials of degree n in k variables is

$$\binom{n+k-1}{k-1} - \binom{n+k-3}{k-1} = \binom{n+k-3}{k-3}\left(\frac{2n}{k-2}+1\right),$$

which is the number of polynomials (1).

3. It is worth noting that the polynomials obtained from (1) by deleting the factor $(-1^{[a_1/2]}$ are solutions of the generalized wave equation $\sum_{j=2}^k \partial^2 u/\partial x_j^2 = \partial^2 u/\partial x_1^2$, which form a basic set for that equation.

4. Further, for each set of k non-negative integers b_j such that $\sum_{j=1}^k b_j = n$, $b_1 \leqq s-1$, the polynomials

$$H^{n,s}_{b_1,b_2,\cdots,b_k}(x_1, x_2, \cdots, x_k) = \sum (-1)^{[a_1/s]} \frac{n!}{\prod_{j=1}^k a_j!} \frac{\left[\frac{a_1}{s}\right]!}{\prod_{j=2}^k \left(\frac{b_j - a_j}{s}\right)!} \prod_{j=1}^k x_j^{a_j},$$

where the summation extends over all a_j such that

(1) $a_j \equiv b_j \bmod s$, $j = 1, 2, \cdots, k$,

(2) $\sum_{j=1}^k a_j = n$,

(3) $a_j \leqq b_j$, $j = 2, 3, \cdots, k$,

provide a basic set of solutions for

$$\sum_{j=1}^k \frac{\partial^s u}{\partial x_j^s} = 0.$$

5. Of particular interest for harmonic polynomials is the case $k=3$. Whittaker[1] has obtained the general solution of $\nabla^2 U(x, y, z) = 0$ by means of an integral. Ketchum[2] gives another form of the general solution as an analytic function of a hypervariable w such that the $2n+1$ linearly independent components of w^n form a basic set of homogeneous harmonic polynomials of degree n. Both of these re-

[1] Math. Ann. vol. 57 (1903) p. 333.

[2] Amer. J. Math. vol. 51 (1929) p. 179.

sults use trigonometric functions, and neither of them displays immediately a set of polynomial solutions of degree n. Morse and Feshbach[3] indicate how one obtains, from a special case of Whittaker's integral, a basic set of degree n, but carry the computation only as far as $n=3$. Courant and Hilbert[4] give a basic set with only one of the $2n+1$ members having real coefficients. The polynomials (1) for $k=3$, $x_1=x$, $x_2=y$ and $x_3=z$, unlike those from the basic sets referred to above, are given explicitly for each n. Thus, for $n=6$ the 13 independent spherical harmonics are obtained by assigning $(b_1\ b_2\ b_3)$ the values (0 6 0), (0 5 1), (0 4 2), (0 3 3), (0 2 4), (0 1 5), (0 0 6), (1 5 0), (1 4 1) (1 3 2), (1 2 3), (1 1 4), and (1 0 5) in turn. We display a typical member,

$$H_{123}^{6} = 60xy^{2}z^{3} - 60x^{3}y^{2}z - 20x^{3}z^{3} + 12x^{5}z.$$

6. The authors wish to express their appreciation to Professor Ernest Ikenberry who directed their attention to the 3-dimensional basic sets given by Morse-Feshbach and Courant-Hilbert and to the referee who pointed out the construction of basic sets of $p+2$ dimensions appearing in *Higher transcendental functions*,[5] A. Erdélyi, editor. These sets for $p+2$ dimensions differ from those of the authors in that the coefficients are in general complex while those of the authors are real.

Added in proof. When these results for $k=3$ only were presented at the International Congress of Mathematicians, September, 1954, Professors P. C. Rosenbloom and L. Bers kindly called the authors' attention to a three variable basic set of harmonic polynomials given by M. H. Protter [*Generalized spherical harmonics*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 314–341]. In a forthcoming note in the Proceedings, the authors point out that their results for $k=3$ give a single formulation for the four classes into which Protter's basic set was divided.

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[3] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, New York, McGraw-Hill, 1953, Part II, pp. 1270–1271.

[4] R. Courant and D. Hilbert, *Methods of mathematical physics*, 1st English ed., New York, Interscience, 1953, vol. I, p. 540.

[5] Arthur Erdélyi, et al., *Higher transcendental functions*, vol. II, (Bateman Manuscript Project) New York, McGraw-Hill, 1953, p. 239.