ON THE ZEROS OF SUCCESSIVE DERIVATIVES

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1. **Statement of the main result.** The following analogue of a theorem of Landau [1, p. 102] seems to be new.

THEOREM 1a. *Consider the class $\mathcal{J}_\alpha$ of all analytic functions f, regular at the origin and such that*

(1.1a) $$1 = f(0) = f'(0); \quad \alpha = f''(0) \neq 1.$$

There exists a positive function $R_1(\alpha)$, finite for

(1.2a) $$\alpha \neq 1,$$

and such that every function $f(\in \mathcal{J}_\alpha)$ has, in the region

(1.3a) $$|z| \leq R_1(\alpha),$$

either a singularity or else a zero of

(1.4a) $$f(z)f'(z)f''(z).$$

The function $R_1(\alpha)$ may be chosen as follows:

(1.5a) $$R_1(\alpha) = \max \left\{ A_1, \frac{A_2}{|\alpha - 1|} \log \frac{1}{|\alpha - 1|} \right\} \quad for \ |\alpha| \leq A_3,$$

(1.6a) $$R_1(\alpha) = A_4 \frac{\log |\alpha|}{|\alpha|} \quad for \ |\alpha| > A_3,$$

where A_1, A_2, A_3, A_4 denote suitable numerical constants (independent of f).

Instead of proving Theorem 1a, we shall prove the following closely related result.

THEOREM 1b. *Consider the class G_β of all analytic functions g, regular at the origin and such that*

(1.1b) $$1 = g(0); \quad \beta = g'(0) \neq 0.$$

There exists a positive function $R_2(\beta)$, finite for

(1.2b) $$\beta \neq 0,$$

and such that every function $g(\in G_\beta)$ has, in the region

(1.3b) $$|z| \leqq R_2(\beta),$$

either a singularity or else a zero of

(1.4b) $$g(z)[g'(z) - 1].$$

  The function $R_2(\beta)$ may be chosen as follows:

(1.5b) $$R_2(\beta) = \max\left[B_1, \frac{B_2}{|\beta|}\log\frac{1}{|\beta|}\right] \qquad for \ |\beta| \leqq B_3,$$

(1.6b) $$R_2(\beta) = B_4\frac{\log|\beta|}{|\beta|} \qquad for \ |\beta| > B_3,$$

where B_1, B_2, B_3, B_4 denote suitable numerical constants (independent of g).

  Theorem 1a is immediately obtained by applying Theorem 1b to the function

$$g(z) = \frac{f(z)}{f'(z)}.$$

  2. Remarks on the structure of Theorem 1a. Our definitions imply that if $f \in \mathcal{F}_\alpha$, the expansion

$$f(z) = 1 + z + \frac{\alpha}{2!}z^2 + \cdots,$$

has a positive (or infinite) radius of convergence $\rho[f]$.

  Now let $R[f]$ be a non-negative number defined as follows:

  (i) $R[f] = \rho[f]$ if $f(z)f'(z)f''(z) \neq 0$ for $|z| < \rho[f]$;

  (ii) *otherwise, let z_0 denote the zero (or one of the zeros) of least modulus of the function $f(z)f'(z)f''(z)$ and put*

$$R[f] = |z_0|.$$

For certain special choices of f it is possible to determine $R[f]$ explicitly. For instance, if $\alpha \neq 1$,

(2.1) $$R\left[\exp\left\{\frac{e^{(\alpha-1)z} - 1}{\alpha - 1}\right\}\right] = \left|\frac{\log(1 - \alpha)}{1 - \alpha}\right|,$$

where the determination of the logarithm is chosen so as to minimize $|\log(1-\alpha)|$.

  If $\alpha \to 1$, (2.1) takes the limiting form

(2.2) $$R[e^z] = +\infty,$$

which is obviously correct. In connection with (2.2), we observe that if $h(z)$ is entire and if $hh'h''$ never vanishes, Theorem 1a shows that h must satisfy the differential equation $hh'' = \{h'\}^2$. Hence, as was proved by Pólya [3, p. 38, footnote 1a] and Saxer [5, p. 206], $h(z)$ is of the form $\exp{(az+b)}$.

Now let $\alpha(\neq 1)$, be fixed. In view of Theorem 1a, the relation

$$(2.3) \qquad\qquad R_0(\alpha) = \sup_{f \in \mathcal{F}_\alpha} R[f],$$

defines a finite quantity $R_0(\alpha)$; moreover, in view of (2.3), (2.1), and (1.3a)

$$(2.4) \qquad\qquad \left| \frac{\log{(1-\alpha)}}{1-\alpha} \right| \leqq R_0(\alpha) \leqq R_1(\alpha).$$

It would be of interest to determine exactly $R_0(\alpha)$. Although this is perhaps difficult, a suitable modification of our arguments readily yields

$$\limsup_{\alpha \to 1} \frac{R_0(\alpha)\,|1-\alpha|}{\left|\log|1-\alpha|\right|} \leqq 2; \qquad \limsup_{\alpha \to \infty} \frac{R_0(\alpha)\,|\alpha|}{\log|\alpha|} \leqq 2.$$

3. **Applications of Theorem** 1a. The limit points of the zeros of the derivatives of an analytic function are usually difficult to investigate. The two theorems which we state and prove in this section will show how Theorem 1a may be used in such investigations.

Let z_0 be a point of regularity of $h(z)$ and assume

$$(3.1) \qquad h^{(n)}(z_0)h^{(n+1)}(z_0) \neq 0, \qquad \alpha = \frac{h^{(n)}(z_0)h^{(n+2)}(z_0)}{\{h^{(n+1)}(z_0)\}^2} \neq 1.$$

Put

$$t = (z - z_0)\frac{h^{(n+1)}(z_0)}{h^{(n)}(z_0)},$$

and consider the function

$$\phi(t) = \frac{h^{(n)}(z_0 + t(h^{(n)}(z_0)/h^{(n+1)}(z_0)))}{h^{(n)}(z_0)} = 1 + t + \alpha\frac{t^2}{2!} + \cdots.$$

Applying Theorem 1a to this function, we obtain bounds for the radius of the largest circle, with center at z_0, free of singularities of $h(z)$ and of zeros of

$$h^{(n)}(z)h^{(n+1)}(z)h^{(n+2)}(z).$$

These considerations obviously yield:

THEOREM 2. *Let $h(z)$ be regular at z_0. Consider a sequence $\{n_\nu\}_{\nu=1}^{\infty}$ of positive integers and the two sequences*

$$(3.2) \qquad \left\{\frac{h^{(n_\nu+1)}(z_0)}{h^{(n_\nu)}(z_0)}\right\}_{\nu=1}^{\infty},$$

$$(3.3) \qquad \left\{\frac{h^{(n_\nu+2)}(z_0)\,h^{(n_\nu)}(z_0)}{[h^{(n_\nu+1)}(z_0)]^2}\right\}_{\nu=1}^{\infty}.$$

The point z_0 is a limit point of the zeros of the derivatives of $h(z)$ if one of the two following conditions is satisfied:

I. *(3.2) tends to infinity and (3.3), if it converges, does not tend to one.*

II. *(3.3) tends to infinity and (3.2), if it converges, does not tend to zero.*

Theorem 2 has the following consequence, which generalizes substantially a result of Rådström [4, p. 138].

THEOREM 3. *Let*

$$h^*(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

be a function which, if it is entire, is not of exponential type.

It is possible to find a sequence $\{\omega_\nu\}_{\nu=0}^{\infty}$ of numbers, each of which is $+1$ or -1 and such that the origin is a limit point of the zeros of the derivatives of

$$h(z) = \sum_{\nu=0}^{\infty} a_\nu \omega_\nu z^\nu.$$

The proof is immediate: we apply Theorem 2 with $z_0 = 0$. As $h^*(z)$ is not of exponential type there exists a sequence $\{n_\nu\}$ such that (3.2) tends to infinity. Moreover, it is clear that a suitable choice of $\{\omega_\nu\}$ will prevent (3.3) from converging to 1.

4. Proof of Theorem 1b. A theorem of Miranda [2, p. 185] contains[1] the following proposition which we state as

LEMMA 1. *Let $h(z)$ be regular for $|z| < 1$ and such that*

$$h(z) \neq 0, \qquad h'(z) \neq 1.$$

Then, there exist two positive absolute constants P and Q such that

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[1] Cf. Valiron [6, p. 21].

(4.1) $$m(1/2, h) < P \log^+ \big| h(0) \big| + Q.$$

Using Nevanlinna's formula [6, p. 12] and Jensen's theorem, (4.1) implies

(4.2)
$$\left| \frac{h'(0)}{h(0)} \right| \leqq 4[m(1/2, h) + m(1/2, 1/h)]$$
$$< 4[2P \log^+ \big| h(0) \big| + 2Q - \log \big| h(0) \big| \,].$$

Let $g(z)$ be a function of the class G_β and assume that for $|z| < R$, $g(z)$ is regular and

$$g(z) \neq 0, \qquad g'(z) \neq 1.$$

Taking

$$h(z) = \frac{g(Rz)}{R},$$

(4.2) yields

(4.3) $$R \big| \beta \big| < 8P \log^+ \frac{1}{R} + 4 \log R + 8Q.$$

We may, if necessary, increase the value of Q so that it is no restriction to assume $Q \geqq 2$. Then, if

(4.4) $$\big| \beta \big| > 8Q^2 (\geqq 32),$$
(4.5) $$R \geqq 1,$$

(4.3), (4.4), and (4.5) imply

(4.6) $$8Q^2 R < 4R^{1/2} + 8Q.$$

As (4.6) contradicts (4.5), it is clear that (4.4) implies $R < 1$ and hence (4.3) yields

$$R \big| \beta \big| + 8P \log R < 8Q.$$

Consider the quantity S uniquely defined by

$$S \big| \beta \big| + 8P \log S = 8Q.$$

Clearly

$$R < S < 1, \qquad S \big| \beta \big| > 8Q,$$

$$R < S < \frac{8Q}{\big| \beta \big|} + \frac{8P}{\big| \beta \big|} \log \frac{\big| \beta \big|}{8Q} < \frac{8(P+Q)}{\big| \beta \big|} \log \big| \beta \big|,$$

so that we may take

$$B_3 = 8Q^2, \qquad B_4 = 8(P + Q).$$

We now find an upper bound for R which does not depend on the assumption

$$|\beta| > B_3.$$

We either have $R < e$ or else (4.3) implies

$$R|\beta| < (4 + 8Q) \log R < (4 + 8Q)R^{1/2},$$

$$R < \frac{4 + 8Q}{|\beta|} \log \left(\frac{4 + 8Q}{|\beta|} \right)^2.$$

The determination of suitable constants B_1 and B_2 is now trivial.

References

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