TWO-DIMENSIONAL MEASURE IN 3-SPACE

GERALD FREILICH

1. **Introduction.** The problem of determining whether a particular measure satisfies the "cylinder property" has been attacked by various mathematicians, using different techniques. Roughly speaking, the "cylinder property" is the intuitively desired property of having the area of a right cylinder constructed over a one-dimensional set in the $x-y$ plane equal to the height times the length of the base. (For a fuller discussion of this problem, see [1].)[1] The first consideration of this problem was in a paper by J. F. Randolph [2]. In his paper, Randolph proved that if we use the two-dimensional Carathéodory measure C_3^2 for area in 3-space and the one-dimensional Carathéodory measure C_2^1 for length in the $x-y$ plane, then the cylinder property is true in one direction. Specifically, if A is a C_2^1-measurable set in the $x-y$ plane, and $A\times h=\{(x, y, z)\mid(x, y)\in A \text{ and } 0\leqq z\leqq h\}$, then

$$h\cdot C_2^1(A) \geqq C_3^2(A \times h).$$

(The proof in [2, p. 270] is slightly incorrect in that Randolph tacitly assumes that a plane convex set can be included in a circle of the same diameter; however his proof is easily corrected by an obvious modification.) We shall prove a partial result in the other direction, namely

$$C_3^2(A \times h) \geqq (h/2)C_2^1(A).$$

This proof leads naturally to a new definition of a two-dimensional measure in 3-space, this being the first measure for which it is possible to prove the cylinder property when length is taken in the sense of Carathéodory (or equivalently Hausdorff).

2. We shall use the notation and definitions of [1]. In particular, $(f:g)$ is the superposition function of f and g, p_n^m is the projection from E^n to E^m defined by

$$p_n^m(x_1, \cdots, x_m, \cdots, x_n) = (x_1, \cdots, x_m),$$

G_n is the orthogonal group of E^n, and $\mathcal{L}_n$ is Lebesgue n-dimensional measure.

We first prove the following

LEMMA. *If d is a bounded convex set in E^3, then*

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[1] Symbols in brackets refer to the bibliography at the end of the paper.

$$\sup_{S\in G_3} \mathcal{L}_2[(\overset{2}{p_3}:S)(d)]$$

$$\geqq \frac{1}{2}\int_{-\infty}^{\infty} \sup_{R\in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(\{(x, y) \mid (x, y, z) \in d\})]dz.$$

PROOF. We first notice that the integrand in the statement of the lemma is a continuous function of z in the interval for which it is $\neq 0$, thus assuring the existence of the integral. Since d is bounded, let W be the supremum of the integrand. Let

$$H = \sup \{ \mid z_1 - z_2 \mid \mid (x_1, y_1, z_1) \in d, (x_2, y_2, z_2) \in d \text{ for some } x_1, y_1, x_2, y_2\},$$

i.e., H is the height of d. Then

$$\int_{-\infty}^{\infty} \sup_{R\in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(\{(x, y) \mid (x, y, z) \in d\})]dz \leqq WH.$$

Now there exists a projection of d into a vertical plane which contains a quadrilateral of height H with a horizontal diagonal of length W. Since the area of such a quadrilateral is $WH/2$, it follows that $\sup_{S\in G_3} \mathcal{L}_2[(\overset{2}{p_3}:S)(d)] \geqq WH/2$. The proof is complete.

THEOREM 1. *If A is a $\overset{1}{C_2}$-measurable subset of E^2 with $\overset{1}{C_2}(A) < \infty$, and $h > 0$, then*

$$\overset{2}{C_3}(A \times h) \geqq (h/2)\cdot \overset{1}{C_2}(A).$$

PROOF. Let $\epsilon > 0$, and choose $\delta > 0$ so small that if $\mathcal{C}$ is a countable covering of A by open convex sets of diameter less than δ, then

$$\sum_{c\in\mathcal{C}} \sup_{R\in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(c)] \geqq \overset{1}{C_2}(A) - \epsilon.$$

Next choose a countable covering $\mathcal{D}$ of $A \times h$ by open convex sets of diameter less than δ and such that

$$\sum_{d\in\mathcal{D}} \sup_{S\in G_3} \mathcal{L}_2[(\overset{2}{p_3}:S)(d)] \leqq \overset{2}{C_3}(A \times h) + \epsilon.$$

If $0 \leqq z \leqq h$, then $\mathcal{C}_z = \{c \mid c = \{(x, y) \mid (x, y, z) \in d\}$ for some $d \in \mathcal{D}\}$ is a countable covering of A by open convex sets of diameter less than δ and therefore

$$\sum_{c\in\mathcal{C}_z} \sup_{R\in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(c)] \geqq \overset{1}{C_2}(A) - \epsilon.$$

Hence using the lemma,

$$
*\left\{
\begin{aligned}
&2\overset{2}{C_3}(A \times h) + 2\epsilon \\
&\quad \geqq \sum_{d \in \mathcal{D}} 2 \sup_{S \in G_3} \mathcal{L}_2[(\overset{2}{p_3}:S)(d)] \\
&\quad \geqq \sum_{d \in \mathcal{D}} \int_0^h \sup_{R \in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(\{(x, y) \mid (x, y, z) \in d\})]dz \\
&\quad = \int_0^h \sum_{d \in \mathcal{D}} \sup_{R \in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(\{(x, y) \mid (x, y, z) \in d\})]dz \\
&\quad \geqq \int_0^h \sum_{c \in \mathcal{C}_z} \sup_{R \in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(c)]dz \\
&\quad \geqq \int_0^h (\overset{1}{C_2}(A) - \epsilon)dz = h \cdot \overset{1}{C_2}(A) - h\epsilon.
\end{aligned}
\right.
$$

Letting ϵ go to zero, we conclude that $2C_3^2(A \times h) \geqq h \cdot C_2^1(A)$. The proof is complete.

3. The proof of §2 leads one naturally to a new definition of a two-dimensional measure in E^3 which we shall denote by P_3^2. For an open convex set d in E^3, define the (gauge) function $P(d)$ by the formula

$$P(d) = \sup_{S \in G_3} \int_{-\infty}^{\infty} \sup_{R \in G_2} \mathcal{L}_1[(\overset{1}{p_2}:R)(\{(x, y) \mid (x, y, z) \in S(d)\})]dz.$$

Then P generates a measure which we shall call P_3^2. Specifically, if B is a subset of E^3, and $r > 0$, define $P_r(B)$ as the infimum of sums of the form $\sum_{d \in \mathcal{D}} P(d)$, where $\mathcal{D}$ is a countable covering of B by convex open sets of diameter less than r. Then $P_3^2(B) = \lim_{r \to 0+} P_r(B)$. Though this definition can be generalized to define P_n^m for all $1 < m < n$, it is not implausible to define $P_n^1 = C_n^1$ and $P_n^n = C_n^n = \mathcal{L}_n$. It then follows that P has the cylinder property, for we can prove

THEOREM 2. *If the conditions of Theorem* 1 *hold, then*

$$h \cdot \overset{1}{P_2}(A) = h \cdot \overset{1}{C_2}(A) = \overset{2}{P_3}(A \times h).$$

SKETCH OF PROOF. Using the proof of Theorem 1, and especially the latter part of the starred chain of inequalities, it follows that $P_3^2(A \times h) \geqq h \cdot C_2^1(A)$. Also the proof used by Randolph in [2] will carry over to show that $P_3^2(A \times h) \leqq h \cdot C_2^1(A)$.

BIBLIOGRAPHY

1. G. Freilich, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 232–275.
2. J. F. Randolph, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 268–274.

CITY COLLEGE, NEW YORK