COMPACT TRANSFORMATIONS AND THE k-TOPOLOGY IN HILBERT SPACE[1]

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1. **Introduction.** We shall be concerned only with Hilbert spaces, though many of the notions involved can be extended to—and indeed sometimes originally appeared in—a wider context. By a Hilbert space H will be meant an inner product space of arbitrary dimension, which is moreover complete. The *k-topology* for H, originally defined in [1],[2] is generated from a basis set of neighborhoods of the identity θ, $\{V(\theta)\}$, obtained in the following way: If K is any compact subset of H, let $V(\theta)=\{x\in H \mid |(x, y)| \leqq 1 \text{ for all } y\in K\}$. The interest of the k-topology lies in the fact that it is the strongest locally convex topology for H which coincides with the weak topology on all spheres [2].

The purpose of this paper is to introduce an equivalent method of defining this topology, the equivalence to be proved via a lemma concerning compact transformations on H to H.

2. **Compact transformations and sets in H.** Lemma 1 is probably well known, being an explicit form of some more general theorems concerning pointwise and uniform convergence of sequences of continuous mappings of a compact set.

LEMMA 1. *Let H be a Hilbert space with a denumerable orthonormal base $(e_1, e_2, \cdots, e_n, \cdots)$ in terms of which every element $x\in H$ has the unique expansion $x=\sum_{i=1}^{\infty} a_i(x)e_i$. Let K be a closed bounded subset of H. Then K is compact if and only if the following criterion holds: For every real $\epsilon>0$, there exists an integer $N(\epsilon)$ such that $\left\| \sum_{i=N+1}^{\infty} a_i(x)e_i\right\| <\epsilon$ for all x in K.*

A linear transformation $T:H\to H$ is called *compact* if $\mathrm{Cl}(T(S))$, the closure of $T(S)$, is compact, where S is the unit ball $\{x\in H \mid \|x\| \leqq 1\}$.

LEMMA 2. *Let H be any Hilbert space, and K any compact subset. Then there exists a compact linear transformation T such that $T(S)\supset K$.*

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

PROOF. We may assume without loss of generality that (1) K is closed, convex, symmetric, and (2) that $K\subset S$. By symmetric is meant that if $x\in K$, and if a is a scalar such that $|a|=1$, then $ax\in K$. (1) is possible because if K is extended to its least convex, symmetric, closed hull, the result is still compact. Proving the lemma for this hull proves it a fortiori for K. (2) follows from the fact that a scalar multiple of a compact transformation is again compact.

Let $a_1=\sup\{\|x\| \mid x\in K\}$. Since K is compact, and $\|x\|$ is a continuous function on K, there is a vector $y_1\in K$ at which the norm a_1 is taken on. Set $e_1=y_1/a_1$. Then e_1 is the first element of an orthonormal sequence constructed inductively as follows: Denote by V_n the linear extension of $\{e_1\cup e_2\cup\cdots\cup e_n\}$, and by $V_n^\perp$ the orthogonal complement of V_n in H. For any element $x\in K$, there is an unique decomposition $x=x^n+u^n$, with $x^n\in V_n$ and $u^n\in V_n^\perp$. Then set $a_{n+1}=\sup\{\|u^n\| \mid x\in K, x=x^n+u^n\}$. Since u^n and hence $\|u^n\|$ is a continuous function on the compact K, there exists a vector $x=x^n+u^n$ at which this supremum is taken on. Let $y_{n+1}=u^n$ for this x and set $e_{n+1}=y_{n+1}/a_{n+1}$, where a_{n+1} is the supremum in question. Clearly the vectors y_n, and hence the unit vectors e_n, form an orthogonal set, with y_{n+1} and hence e_{n+1} orthogonal to V_n. It is also clear from the construction that $a_{n+1}\leqq a_n$ for all n. Moreover, $\lim_n a_n=0$, for if this were not the case, there is some $\delta>0$ such that $a_n>\delta$ for all n. But this is to say that in the construction, if x_n is the sequence in K at which the successive maxima are taken on, and if we represent $x_n=x^n+u^n$ as above, then $\|u^n\|>\delta$ for all n. Then for $n>m$, $\|x_m-x_n\|=\|x^m-x^n+u^m-u^n\|\geqq\|u^n\|>\delta$, because the sum of the first three terms is in V_n while the fourth is orthogonal to V_n. Hence the sequence $\{x_n\}$ has no point of accumulation, denying the compactness of K.

Let $\mathcal{H}$ be the closure of $\bigcup_1^\infty V_n$. $\mathcal{H}$ is complete and separable, i.e. it has a countable orthonormal base, the set $\{e_n\}$ in fact. Further, $\mathcal{H}\supset K$. For, if $y\in K$, $y=y^n+u^n$, with $y^n\in V_n\subset\mathcal{H}$, and $\|u^n\|\leqq a_{n+1}$ which becomes arbitrarily small with increasing n. Hence y is arbitrarily close to $\bigcup_1^\infty V_n$, the closure of which is $\mathcal{H}$. Thus, for any $x\in K$, $x=\sum_1^\infty x_ie_i$, and $\|\sum_n^\infty x_ie_i\|=(\sum_n^\infty |x_i|^2)^{1/2}\leqq a_n$ for all n. We also observe, by assumption (2) of the opening of this proof, that $a_1\leqq 1$.

Now let $b_n=(2a_n)^{1/2}$. Clearly $b_n\geqq(a_n)^{1/2}\geqq a_n$, since $a_n\leqq 1$. Also, $b_n\geqq b_{n+1}$ for all n, and $\lim_n b_n=0$. We construct T as follows: Let $Te_i=b_ie_i$ for all i, and extend T by linearity and continuity to all of $\mathcal{H}$. For $x\in\mathcal{H}^\perp$, define $Tx=\theta$, and again T may be extended by linearity, this time to all of H.

To show T is a compact transformation, it will suffice to apply

Lemma 1 to the set $\mathrm{Cl}(T(S))$. Let $\epsilon>0$ be given, and choose N such that $b_N<\epsilon/2$. Now if $y=\sum_1^\infty y_ie_i\in\mathrm{Cl}(T(S))$, there exists some $x=\sum_1^\infty x_ie_i\in T(S)$ such that $\|x-y\|<\epsilon/2$. Further, there exists some $z=\sum_1^\infty z_ie_i\in S$ such that $Tz=x$, i.e. $\sum_1^\infty z_ib_ie_i=\sum_1^\infty x_ie_i$. Then $\|\sum_N^\infty y_ie_i\|\leqq\|\sum_N^\infty (y_i-x_i)e_i+\sum_N^\infty x_ie_i\|\leqq\|y-x\|+\|\sum_N^\infty b_iz_ie_i\|<\epsilon/2+b_N\|z\|<\epsilon$. Thus the criterion of Lemma 1 is fulfilled.

It only remains to be shown that $T(S)\supset K$. To this end, let $x=\sum_1^\infty x_ie_i\in K$. Then if $z=\sum_1^\infty (x_i/b_i)e_i$, $Tz=x$. To show $z\in S$ completes the proof. We shall show $\|z\|^2\leqq 1$ by showing that

$$\sum_1^n (|x_i|^2/b_i^2)\leqq 1$$

for all n. If we set $R_n=\sum_n^\infty |x_i|^2$, and observe that $R_n\leqq a_n^2$, then

$$\begin{aligned}
\sum_1^n (|x_i|^2/b_i^2) &= \sum_1^n \frac{R_i-R_{i+1}}{2a_i}\\
&= \frac{1}{2}\left[\sum_0^{n-1}\frac{R_{i+1}}{a_{i+1}}-\sum_1^n\frac{R_{i+1}}{a_i}\right]\\
&= \frac{1}{2}\left[\left(\frac{R_1}{a_1}-\frac{R_{n+1}}{a_n}\right)+\sum_1^{n-1}R_{i+1}\left(\frac{1}{a_{i+1}}-\frac{1}{a_i}\right)\right]\\
&\leqq \frac{1}{2}\left[a_1+\sum_1^{n-1}a_{i+1}^2\left(\frac{a_i-a_{i+1}}{a_{i+1}a_i}\right)\right]\\
&= \frac{1}{2}\left[a_1+\sum_1^{n-1}\frac{a_{i+1}}{a_i}(a_i-a_{i+1})\right]\\
&\leqq \frac{1}{2}\left[a_1+\sum_1^{n-1}(a_i-a_{i+1})\right]\\
&= \frac{1}{2}[a_1+a_1-a_n]\\
&\leqq a_1\leqq 1.
\end{aligned}$$

Q.E.D.

3. **Alternate definition of the k-topology.** Let $C(H, H)$ denote the class of all compact linear transformations, and let the *c-topology* be defined as follows: The typical neighborhood of θ, $V(\theta)=\{x\in H \mid \|T_ix\|\leqq 1, T_i\in C(H, H), i=1, 2, \cdots, n\}$. A basis of c-neighborhoods of θ is obtained by letting the finite sets $\{T_i\}$ run through the class of all finite subsets of $C(H, H)$. That a locally convex topology is thus produced for H is easily verified directly; indeed the c-topology is nothing but the point-open topology on H, when H is regarded as a space of functions from $C(H, H)$ to H, under the rule $x: T\rightarrow Tx$.

THEOREM. *The topologies c and k are identical.*

PROOF. Let $V(\theta)$ be any k-neighborhood: $V(\theta)=\{x\in H \mid |(x, y)| \leqq 1$ for all $y\in K$, K compact$\}$. Let $T\in C(H, H)$ be chosen (according to Lemma 1) so that $T(S)\supset K$, and let T' be the adjoint of T. As is well known, $T'\in C(H, H)$ also. If W is the c-neighborhood $\{x\in H \mid \|T'x\| \leqq 1\}$, then $W\subset V$, for let $x\in W$. Then $\|T'x\|\leqq 1$, and if $u\in S$, $|(u, T'x)|\leqq 1$. Thus $|(Tu, x)|\leqq 1$ for all $u\in S$, and a fortiori, $|(y, x)|\leqq 1$ for all $y\in K$. Conversely, let $W(\theta)$ be a c-neighborhood: $W=\{x\in H \mid \|T_i x\|\leqq 1,\ T_i\in C(H, H),\ i=1, 2, \cdots, n\}$. Then for all $u\in S$, $|(u, T_i x)|\leqq 1$, and $|(T_i' u, x)|\leqq 1$, for each i. The T_i' are all in $C(H, H)$. Let $K=\bigcup_{i=1}^{n} \mathrm{Cl}(T_i'(S))$. Then the k-neighborhood $V(\theta)=\{x\in H \mid |(y, x)|\leqq 1$ for all $y\in K\}$ clearly satisfies $V(\theta)\subset W(\theta)$. Q.E.D.

The definitions for the k- and c-topologies can be generalized considerably, but even for complete normed spaces having a Schauder base it is not possible to prove Lemma 2 by the device used above. However, a statement somewhat weaker than Lemma 2 would suffice: that given a compact set K, there exist a finite number of compact transformations T_i such that the union of $T_i(S)$ contain K.

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