ON TWO PROBLEMS OF J. SCHMIDT

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Two problems are raised by J. Schmidt in one of his papers [1]. We shall give solutions, but first introduce certain concepts and definitions on which these problems are based in J. Schmidt's article.

At the outset an arbitrary nonempty set E is fixed.

Denoting the class of all subsets of E by $\mathcal{S}(E)$, we shall say that a subclass K of $\mathcal{S}(E)$ is a *closure class* on E , if

- (i) $E \in K$,
- (ii) the intersection $\bigcap_{T \in \mathcal{T}} T$ of every nonempty subclass $\mathcal{T}$ of K is an element of K .

Elements of K will be called K -closed or, briefly, *closed*. A closure class K on E is on occasion also denoted by $[E, K]$.

If K is a closure class on E , and M an arbitrary subset of E , then we mean by the K -closure (or, briefly, *closure*) of M the set

$$C_K(M) = \bigcap_{M \subseteq K \in \mathcal{K}} K. \quad (\text{Briefly, } C(M)).$$

It follows from the definition of closure class that $C(M)$ is closed.

The function $C = C_K$, defined on $\mathcal{S}(E)$, with values in $\mathcal{S}(E)$, is called a *closure operator* on E . It follows from the definitions that every closure operator C satisfies:

- C1. $M_1 \subseteq M_2 \rightarrow C(M_1) \subseteq C(M_2)$,
- C2. $M \subseteq C(M)$,
- C3. $C(C(M)) = C(M)$.

Every set-valued set function on $\mathcal{S}(E)$ satisfying C1, C2, and C3 determines a closure class, namely its range, and this correspondence between closure classes on E and closure operators on E is one to one. Two closure operators are called equal, $C_1 = C_2$, if they have the same range.

$\mathcal{T}$ is called a *chain* in $\mathcal{M} \subseteq \mathcal{S}(E)$ if $\mathcal{T} \subseteq \mathcal{M}$ and if $\mathcal{T}$ is simply ordered by the relation $\subseteq$.

A subclass $\mathcal{M}$ of $\mathcal{S}(E)$ is called *inductive* if the union of every non-empty chain in $\mathcal{M}$ is an element of $\mathcal{M}$.

A subclass $\mathcal{M}$ of $\mathcal{S}(E)$ is called a *hereditary* class on E , if

$$M \in \mathcal{M} \quad \text{and} \quad M_1 \subset M \rightarrow M_1 \in \mathcal{M}.$$

Obviously, $\theta_E \in \mathcal{M}$, when θ_E is the empty set and $\mathcal{M}$ a hereditary class on E .

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If K is the closure class corresponding to $C = C\mathcal{K}$, then a subset M of E is called *C-independent* (or *K-independent*), if

$$M_1 \subseteq M \quad \text{and} \quad C(M_1) = C(M) \rightarrow M_1 = M.$$

Otherwise, M is called *C-dependent* (or *K-dependent*).

It may be easily proved that the class of all independent sets corresponding to a given closure operator C is a hereditary class on E . We denote this class by $\mathfrak{I}_C$ (or, on occasion, $\mathfrak{I}\mathcal{K}$). $\mathfrak{I}_C$ is uniquely determined by C , but it may happen that

$$C_1 \neq C_2 \quad \text{and} \quad \mathfrak{I}_{C_1} = \mathfrak{I}_{C_2}.$$

(This will be illustrated in the sequel.)

It follows from the definitions and from J. Schmidt's study of inductive closure classes in relation to independence that $\mathfrak{I}_C$ is inductive whenever $C = C\mathcal{K}$ corresponds to an inductive closure class K .

It is easy to deduce from relations C1, C2, and C3 the following

LEMMA 1. *If $M \subseteq E$ and $x \in C(M)$, then*

$$C(M) = C(\{x\} \cup M).$$

This lemma will be used later.

STATEMENT OF PROBLEMS GIVEN BY J. SCHMIDT

QUESTION I. May an arbitrarily given nonempty hereditary class of sets be considered as the class of all K -independent sets of a suitable closure class $[E, K]$?

QUESTION II. Suppose that K is a closure class such that the class of all K -independent sets is inductive. Does this imply that K is inductive?

SOLUTIONS

I. The answer to question I is in the negative, by virtue of the following

COUNTEREXAMPLE. Set $E = E_6 = \{1, 2, 3, 4, 5, 6\}$ and consider on E_6 the hereditary class $\mathcal{K}$ which consists of all subsets of the sets

$$P_1 = \{1, 2, 3\}, \quad P_2 = \{3, 4, 5\}, \quad P_3 = \{1, 5, 6\};$$

then there exists no closure operator C on E_6 satisfying: $\mathcal{K} = \mathfrak{I}_C$.

PROOF.¹ Assume: $\exists C$ on $E_6 \ni \mathcal{K} = \mathfrak{I}_C$.

LEMMA 2. *If $\{x, y\} \subset P_i$ ($i = 1, 2$, or 3), then $x \notin C\{y\}$ and $y \notin C\{x\}$.*

For by Lemma 1:

¹ The following proof is due to W. R. Scott.

$$x \in C\{y\} \rightarrow C\{y\} = C\{x, y\},$$

making $\{x, y\}$ C -dependent and contradicting the fact that C -independence is a hereditary property. By the same reasoning: $y \in C\{x\}$, and the lemma is proved.

Consider now the set $\{2, 4, 6\} \subset E_6$. This set, and each of its two-element subsets, must be C -dependent, since they are not contained in any P_i . Since the two-element subsets as well as the individual elements of $\{2, 4, 6\}$ possess a cyclic symmetry with respect to P_1, P_2 , and P_3 , there results no loss of generality from assuming:

$$(i) \quad C\{2, 4, 6\} = C\{2, 4\} = C\{2\}.$$

Again, there is no loss of generality, if, corresponding to our choice (i), we choose that two-element subset $\{4, 6\} \subset \{2, 4, 6\}$ which does not contain $\{2\}$, and if we assume:

$$(ii) \quad C\{4, 6\} = C\{4\}.$$

Corresponding to the choice (ii), we consider next the set $\{1\}$, which is the intersection of the two sets, P_1 and P_3 , which do not have (4) as element. From Lemma 2:

$$(1) \notin C\{2\}, \quad (6) \notin C\{1\}.$$

Now, $\{1, 4\}$ must be C -dependent. If $C\{1, 4\} = C\{4\}$, then $(1) \in C\{4\} \subseteq C\{2, 4\} = C\{2\}$. If $C\{1, 4\} = C\{1\}$, then $C\{1\} \supseteq C\{4\} = C\{4, 6\}$; hence $(6) \in C\{1\}$. Thus, in each of these two cases, a contradiction to Lemma 2 is obtained. From the above-mentioned cyclic symmetry it is obvious that a contradiction of this type will be obtained, regardless of the choice of a two-element set, and of the subsequent choice of a one-element set, by which the required C -dependence of the set $\{2, 4, 6\}$, and of its two-element subsets, may be satisfied. This completes the proof.

II. The answer to question II is in the negative.

PROOF. Let $E = R_1$ be the set of all real numbers, and consider the closure class $\mathcal{K}$ on R_1 whose elements are all topologically closed symmetric intervals with center (0), where the sets R_1 and $\{0\}$ are understood to belong to $\mathcal{K}$.

If $M \subseteq R_1$, then $C\mathcal{K}(M)$ is the least closed interval, centered at (0), which contains M . If M has more than one point: suppose $x \in M$, $y \in M$, $|y| \geq |x|$. Then obviously $x \in C\{y\}$, hence $C\{x, y\} = C\{y\}$. Thus $\{x, y\}$ is C -dependent, and by the heredity property of $\mathfrak{J}_C$, M is C -dependent. The set $\{0\} = \bigcap_{\theta \in \mathcal{K}} \theta \in \mathcal{K} = C(\theta_E)$ is also C -dependent since

$$\{0\} \in \mathcal{K} \rightarrow C\{0\} = \{0\} \rightarrow C(\theta_E) = C\{0\}.$$

Every set $\{x\} \ni (0) \neq x \in R_1$ is independent by:

$$C(\theta_E) = \{0\}, \text{ and } C\{x\} \neq \{0\}.$$

Thus the elements of $\mathfrak{I}_C$ are: the empty set θ_E and the sets $\{x\} \ni (0) \neq x \in R_1$. The chains in $\mathfrak{I}_C$ are:

- (i) the class $\{\theta_E\}$,
- (ii) the classes $\{\{x\} \ni (0) \neq x \in R_1\}$,
- (iii) the classes $\{\theta_E, \{x\} \ni (0) \neq x \in R_1\}$.

Hence $\mathfrak{I}_C$ is inductive. But $\mathcal{K}$ is not inductive, since the union of a chain in $\mathcal{K}$ may be an *open* bounded interval. This completes the proof.

We shall now substantiate an earlier remark, that two different closure operators C_1 and C_2 on E may determine the same class of independent sets, $\mathfrak{I}_{C_1} = \mathfrak{I}_{C_2}$.

Denote the closure operator on R_1 corresponding to the closure class used in the preceding proof by C_1 , and the closure operator on R_1 which corresponds to the closure class $\{R_1, \{0\}\}$ by C_2 .

Then obviously: $C_1 \neq C_2$, since the C_2 -closed sets are just the sets R_1 and $\{0\}$; and clearly $\mathfrak{I}_{C_1} = \mathfrak{I}_{C_2}$.

This example shows also that an inductive closure class may determine the class $\mathfrak{I}_{\mathcal{K}}$ of a noninductive closure class $\mathcal{K}$. For the closure class corresponding to C_2 above is inductive since finite.

BIBLIOGRAPHY

1. Jürgen Schmidt, *Ueber die Rolle der transfiniten Schlussweisen in einer allgemeinen Idealtheorie*, Mathematische Nachrichten vol. 7 (1952) pp. 165–182.

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