ON THE NUMBER OF REALIZATIONS OF A HASSE DIAGRAM BY FINITE SETS

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R. Dedekind proposed the problem, still unsolved for $n > 6$, of finding the order $f(n)$ of the free distributive lattice with n generators.¹ It is known that $f(n)$ is the number of families F of subsets of a set of n objects such that no subset in F includes any other.² A closely related problem is that of finding the number of families of subsets having specified inclusion relations, i.e., a given Hasse diagram.³ For families of two subsets, G. N. Raney has solved this problem in terms of a general n .⁴ The present paper contains the solution, by other methods, for families of four or less subsets. In the process, we illustrate both recursive and direct methods of finding the numbers of realizations of Hasse diagrams by subsets of a finite set.

1. Notation. Let Σ be a set of n objects. We shall use *subset* to mean subset of Σ . By *realization* of a Hasse diagram will be meant a realization by a family $\alpha_1, \dots, \alpha_t$ of subsets. Let t be the number of inclusion relations among the α 's. Families with different Hasse diagrams may have the same s and t , e.g., $s=4$ and $t=5$ for realizations of both

We shall denote by $a_{st}(n)$, $b_{st}(n)$, etc. the numbers of realizations of the respective diagrams with a given s and t . Dedekind's $f(n)$ may be expressed in the form $\sum_{s=1}^{2^n} a_{s0}(n)$.

If $g(n)$ is a function defined for non-negative integers, the sum $\sum_{i=0}^n n C_i g(i)$ will be denoted by $Tg(n)$. Clearly $Tm^n = (m+1)^n$. In the natural way this definition is extended to polynomials, with rational coefficients, in T , operating on functions $g(n)$.

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¹ See [1, p. 147]. Dedekind gave the order for $n \leq 4$. R. Church has the order for $n=5$ in [5] and M. Ward the order for $n=6$ in [6].

² See Theorem 5.14 in [4]. Birkhoff states that this is essentially due to Th. Skolem [2]. See also [3].

³ A Hasse diagram pictures the order relations in a partially ordered finite set. As applied here, each subset is represented by a vertex and an upward slanting segment is drawn from the vertex for α to that for β if $\alpha \subset \beta$ and the family has no γ such that $\alpha \subset \gamma \subset \beta$.

⁴ This verbally communicated result inspired the present paper.

2. A recursive method.

THEOREM I. *Let D be a Hasse diagram with one minimal[§] vertex, V . Let D' be the diagram obtained by deleting from D the vertex V (and the lines from V). Let $a(n)$ and $a'(n)$ be the numbers of realizations of D and D' respectively. Then $a(n) = (T-1)a'(n)$ or $Ta'(n)$, depending on whether D' has one or more than one minimal vertex.*

PROOF. One can choose a subset α of $n-i$ of the n objects in ${}_nC_i$ ways. Let $\beta_1, \dots, \beta_s$ be any of the $a'(i)$ realizations of D' by subsets of the remaining i objects. If D' has at least two minimal vertices, no β_j is null and $\alpha, \alpha \cup \beta_1, \dots, \alpha \cup \beta_s$ is a realization of D . Since each realization of D is obtained uniquely in this way,

$$a(n) = \sum_{i=0}^n {}_nC_i a'(i) = Ta'(n),$$

in this case. If D' has only one minimal vertex, it is clear that $a(n)$ is $Ta'(n)$ minus the number of times, $a'(n)$, in which α coincides with an $\alpha \cup \beta_j$.

Theorem I is generalized in §8 below.

3. The solution for 2 subsets. First let $t=1$. There is just the one diagram

which by Theorem I has $a_{21}(n) = (T-1)2^n = 3^n - 2^n$ realizations.

Since the total number of pairs of subsets is $2^n(2^n-1)/2$, we have for $t=0$, i.e., the diagram

$$a_{20}(n) = \{2^n(2^n-1)/2 - a_{21}(n) = (4^n - 2 \cdot 3^n + 2^n)/2.$$

4. The solution for 3 subsets. The only diagram for $t=3$ is

Using Theorem I, we see that

$$a_{33}(n) = (T-1)a_{21}(n) = 4^n - 2 \cdot 3^n + 2^n.$$

By duality the two diagrams,

[§] A minimal vertex is one from which there are no downward sloping segments.

for $t=2$ have the same number of realizations. Theorem I shows that this is given by

$$a_{32}(n) = Ta_{20}(n) = (5^n - 2 \cdot 4^n + 3^n)/2.$$

There is just the diagram

for $t=1$. We obtain the number, $a_{31}(n)$, of realizations by considering the number, $h(n)$, of ordered triples (α, β, γ) of subsets such that α properly includes β and γ is different from α and from β . Clearly $h(n) = (2^n - 2)a_{21}(n)$. Also $h(n)$ is the number of realizations of

plus three times the number, $a_{33}(n)$, of realizations of

plus two times the number, $2a_{32}(n)$, of realizations of

Hence

$$a_{31}(n) = (2^n - 2)a_{21}(n) - 3a_{33}(n) - 4a_{32}(n) = 6^n - 2 \cdot 5^n + 2 \cdot 3^n - 2^n.$$

Similarly, the number of realizations of the diagram

for $t=0$ is

$$\begin{aligned} a_{30}(n) &= [(2^n - 2)a_{20} - 2a_{32} - 2a_{31}]/3 \\ &= (8^n - 6 \cdot 6^n + 6 \cdot 5^n + 3 \cdot 4^n - 6 \cdot 3^n + 2 \cdot 2^n)/6. \end{aligned}$$

5. The solution for 4 subsets. With the exception of the formulas for $b_{44}(n)$ and $c_{43}(n)$, which are derived in §§9 and 10 respectively, all the material in the following table of solutions for $s=4$ may be found readily by the methods of the previous sections.

6. Two lemmas.

LEMMA 1. *The number of ordered s -tuples $(\sigma_1, \dots, \sigma_s)$ of disjoint subsets is $(s+1)^n = T^s \cdot 1$.*

The essence of the proof is shown by treating the case $s=3$. One

SOLUTIONS FOR $s = 4$

Diagram	t	Number of Realizations
	6	$a_{46} = (T-1)a_{33} = 5^n - 3 \cdot 4^n + 3 \cdot 3^n - 2^n$
	5	$a_{45} = Ta_{32} = (6^n - 2 \cdot 5^n + 4^n)/2$
	5	$b_{45} = (T-1)a_{32} = (6^n - 3 \cdot 5^n + 3 \cdot 4^n - 3^n)/2$
	4	$a_{44} = Ta_{31} = 7^n - 2 \cdot 6^n + 2 \cdot 4^n - 3^n$
	4	$b_{44} = [T(T-1)^2/2]^2 \cdot 1$ $= (7^n - 4 \cdot 6^n + 6 \cdot 5^n - 4 \cdot 4^n + 3^n)/4$
	3	$a_{43} = (2^n - 3)a_{33} - 4a_{46} - 2a_{45} - 4b_{45} - 2a_{44}$ $= 8^n - 2 \cdot 7^n - 6^n + 4 \cdot 5^n - 4^n - 2 \cdot 3^n + 2^n$
	3	$b_{43} = Ta_{30}$ $= (9^n - 6 \cdot 7^n + 6 \cdot 6^n + 3 \cdot 5^n - 6 \cdot 4^n + 2 \cdot 3^n)/6$
	3	$c_{43} = T^4(T-1)^3 \cdot 1 = 8^n - 3 \cdot 7^n + 3 \cdot 6^n - 5^n$
	2	$a_{42} = a_{21}(a_{21} - 1)/2 - 3a_{46} - 2a_{45} - 4b_{45} - 2a_{44} - 2b_{44}$ $- c_{43} - 3a_{33} - 2a_{32}$ $= (9^n - 2 \cdot 8^n + 7^n - 2 \cdot 6^n + 4 \cdot 5^n - 4^n - 2 \cdot 3^n + 2^n)/2$
	2	$b_{42} = (2^n - 3)a_{32} - a_{46} - 2b_{45} - 2a_{44} - 2b_{44} - 3b_{43} - c_{43}$ $= (10^n - 9^n - 4 \cdot 8^n + 7 \cdot 7^n - 2 \cdot 6^n - 2 \cdot 5^n + 4^n)/2$
	1	$a_{41} = [(2^n - 3)a_{31} - 2a_{44} - 3a_{43} - 2c_{43} - 4a_{42} - 4b_{42}]/2$ $= (12^n - 4 \cdot 10^n + 7 \cdot 8^n - 6 \cdot 7^n + 8 \cdot 6^n - 8 \cdot 5^n$ $- 2 \cdot 4^n + 6 \cdot 3^n - 2 \cdot 2^n)/2$
	0	$a_{40} = [(2^n - 3)a_{30} - 2b_{43} - 2b_{42} - 2a_{41}]/4$ $= (16^n - 12 \cdot 12^n + 24 \cdot 10^n + 4 \cdot 9^n - 18 \cdot 8^n + 6 \cdot 7^n$ $- 36 \cdot 6^n + 36 \cdot 5^n + 11 \cdot 4^n - 22 \cdot 3^n + 6 \cdot 2^n)/24$

may choose σ_1 in ${}_nC_i$ ways as a subset with i members. In ${}_{n-i}C_j$ ways, σ_2 may then be chosen as a subset with j members. One chooses σ_3 as any of the 2^{n-i-j} subsets of the remaining $n-i-j$ objects. The total number of such triples is thus

$$\sum_{i=0}^n {}nC_i \sum_{j=0}^{n-i} {}_{n-i}C_j \cdot 2^{n-i-j} = \sum_{i=0}^n 3^{n-i} {}nC_i = 4^n.$$

In a similar fashion one obtains

LEMMA 2. *The number of ordered s -tuples $(\sigma_1, \dots, \sigma_s)$, such that the σ_i are disjoint subsets and $\sigma_1, \dots, \sigma_s$ are not empty is $T^{s-s}(T-1)^s \cdot 1$.*

7. **The direct method.** Let $\alpha_1, \dots, \alpha_s$ be a family of subsets. In the ordered s -tuple $(u_1, \dots, u_s)$ let each u_i be 1 or 0. Let ρ_k , where $k = \sum_{i=1}^s 2^{i-1}u_i$, designate the intersection $\alpha'_1 \cap \alpha'_2 \cap \dots \cap \alpha'_s$, in which α'_i is α_i if $u_i = 1$ and α'_i is the complement in Σ of α_i if $u_i = 0$. It is clear that all families of subsets, for which a given selection of the ρ_k are not empty and the remainder are empty, have the same Hasse diagram. Since the 2^s subsets ρ_k are disjoint, it follows from Lemma 2 that the number of realizations of a diagram D is

$$\frac{1}{m} \sum_{j=0}^{2^s} q_j (T-1)^j \cdot 1,$$

where q_j is the number of divisions, leading to the diagram D , of the ρ_k into j nonempty ones and $2^s - j$ empty ones, and where m is the number of inclusion (and noninclusion) preserving permutations of a family realizing D .

8. **On "products" of diagrams.** The following generalization of Lemma 1 is obtained in the same way as the formula of the previous section.

LEMMA 3. *Let $(D_1, \dots, D_r)$ be an ordered r -tuple of Hasse diagrams. Let D_i have $A_i(T) \cdot 1$ realizations. Then $A_1(T) \cdot \dots \cdot A_r(T) \cdot 1$ is the number of ordered r -tuples $(F_1, \dots, F_r)$ of families of subsets such that F_i is a realization of D_i and each subset in F_i is disjoint with each subset in F_j for $i \neq j$.*

Using Lemma 3, Theorem I is easily generalized as follows:

THEOREM II. *Let D' and D'' be Hasse diagrams. Let D be the diagram formed by placing D' below D'' and drawing segments upward from each maximal vertex of D' to each minimal vertex of D'' . Let $A(T) \cdot 1$, $A'(T) \cdot 1$, and $A''(T) \cdot 1$ be the numbers of realizations of D , D' , and D'' respectively. Then $A(T) = A'(T)A''(T)$ if either D' has more than one maxi-*

mal vertex or D'' has more than one minimal vertex. If there is only one maximal vertex V' of D' and only one minimal vertex V'' of D'' , then $A(T) \cdot 1 = A'(T)A''(T) \cdot 1$ minus the number of realizations of the diagram formed by placing D' and D'' so that V' and V'' coincide.

9. Calculation of b_{44} . Applying Theorem II with D chosen as

and both D' and D'' as

• • ,

we have

$$b_{44}(n) = [T(T-1)^2/2] \cdot 1 = (7^n - 4 \cdot 6^n + 6 \cdot 5^n - 4 \cdot 4^n + 3^n)/4.$$

10. Calculation of c_{43} . Using the notation of §7, we find the number of realizations of

as follows. Examination shows it to be necessary and sufficient for a family $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to have just the inclusion relations $\alpha_1 \subset \alpha_2$, $\alpha_3 \subset \alpha_2$, and $\alpha_3 \subset \alpha_4$ that ρ_k be empty for $k=1, 4, 5, 6, 7, 9, 12, 13$ and not empty for $k=3, 8, 14$.

Let $(\sigma_1, \dots, \sigma_7)$ be an ordered septuple of disjoint subsets such that $\sigma_1, \sigma_2, \sigma_3$ are not empty. A family of α_i realizing

will result if one lets $\rho_3, \rho_8, \rho_{14}, \rho_2, \rho_{10}, \rho_{11}, \rho_{15}$ be $\sigma_1, \dots, \sigma_7$ respectively, lets ρ_0 be the complement in Σ of the union of the σ_j , and chooses the remaining ρ_k to be empty. Since each realization is obtained exactly once in this way, it follows from Lemma 2 that

$$c_{43}(n) = T^4(T-1)^3 \cdot 1 = 8^n - 3 \cdot 7^n + 3 \cdot 6^n - 5^n.$$

11. An observation. In view of the role of the finite difference operator $T-1$ in the above, it is interesting to note that a polynomial $A(T)$, with rational coefficients, is divisible by the v th power of $T-1$ if and only if $a(n) = A(T) \cdot 1$ vanishes for $n=0, 1, \dots, v-1$.

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ON COMPLETE LATTICES AND A PROBLEM OF BIRKHOFF AND FRINK¹

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Two decomposition theorems for elements of complete lattices in terms of join prime or completely join prime elements are obtained (Theorems 1 and 2). The latter theorem gives a solution for a problem of Birkhoff and Frink on the relation between completely prime and completely join irreducible ideals. An interesting special case of this solution—for complemented lattices—is also noted (Theorem 3).

1. Definitions. An element a of a lattice L is called *join irreducible* if $a_1 \cup a_2 = a \rightarrow a_1$ or $a_2 = a$, and *join prime* if $a_1 \cup a_2 \geq a \rightarrow a_1$ or $a_2 \geq a$. Similarly a is called *completely join irreducible* if (for all existing joins $\cup_i a_i$) $\cup_i a_i = a \rightarrow \text{some } a_i = a$, and *completely join prime* if $\cup_i a_i \geq a \rightarrow \text{some } a_i \geq a$.

The definitions of the corresponding dual concepts (indicated in each case by using the prefix "meet" in the place of "join") are quite clear.

A lattice L will be said to be infinitely meet distributive if all (existing) meets $\cap_i a_i$ in it are distributive; the meet $\cap_i a_i$ is said to

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