ON u-STABLE COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

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A commutative power-associative algebra A of characteristic >5 with an idempotent u may be written[1] as the supplementary sum $A=A_u(1)+A_u(1/2)+A_u(0)$ where $A_u(\lambda)$ is the set of all x_λ in A with the property $x_\lambda u=\lambda x_\lambda$. The subspaces $A_u(1)$ and $A_u(0)$ are orthogonal subalgebras, $[A_u(1/2)]^2\subseteq A_u(1)+A_u(0)$ and $A_u(\lambda)A_u(1/2)\subseteq A_u(1/2)+A_u(1-\lambda)$ for $\lambda=0, 1$. The algebra A is called u-stable if $A_u(\lambda)A_u(1/2)\subseteq A_u(1/2)$ and is called stable if it is u-stable for every idempotent element u of A.

A. A. Albert has shown in [**3**] that a simple commutative power-associative algebra A of degree >1 over its center F with characteristic prime to 30 is a Jordan algebra if and only if it is stable. Moreover, it is known that every simple algebra of degree >2 is a Jordan algebra. Thus there remains the problem of determining the nonstable simple algebras of degree two. There do exist simple algebras of characteristic $p>5$ which are not Jordan algebras [**3**; **4**]. Of course, these algebras are not stable, although they may be u-stable for some idempotent u. In this paper we shall obtain the following result.

THEOREM. *Let A be a u-stable simple commutative power-associative algebra of degree 2 over its center F of characteristic zero. Then A is a Jordan algebra.*

We shall use all of the results of [**3**] so we shall adopt the notations of that paper. In particular, all the results of the section giving properties of u-stable algebras will be used. For convenience let us state a few of the required results here.

In a simple u-stable algebra A there exists an element w in $A_u(1/2)$ such that $w^2=1$. Then $A_u(1)=uB$, $A_u(0)=vB$, and $A_u(1/2)=wB+G$ where B is the set of all elements b of $C=A_u(1)+A_u(0)$ with the property $(wb)w=b$ and G is the set of all quantities g of $A_u(1/2)$ with the property $wg=0$. Since $e=(1/2)(1+w)$ and $f=1-e$ are orthogonal idempotents, we may decompose A relative to e. It can be shown that $A_e(1)=eB$, $A_e(0)=fB$, and $A_e(1/2)=B(u-v)+G$. The set B is a subalgebra of C and the product of two elements in G is in B. Also,

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[1] The results of this paragraph are given in [1]. The numbers in brackets refer to the bibliography at the end of the paper.

the following multiplicative relationships exist for any a, b in B, g in G.

$$w(bu) = w(bv) = (1/2)wb, \tag{1}$$

$$(wa)b = w(ab),\ (wa)(wb) = ab, \tag{2}$$

$$g[b(u-v)] = wd, \tag{3}$$

$$gb = h - wc, \tag{4}$$

$$(wb)g + w(gb) = -d(u-v), \tag{5}$$

$$(wb)[a(u-v)] = k, \tag{6}$$

for h, k in G, and c, d in B. The quantity d in relation (5) is the d of (3).

The theorem can evidently be reduced to the case where F is algebraically closed. Then[2] $A_u(1) = uF + N_1$ and $A_u(0) = vF + N_0$ where N_λ is the radical of $A_u(\lambda)$ and $N' = N_1 \oplus N_0 = N + N(u-v)$ is the radical of C where N is the radical of B. Similarly, $A_e(1) = eF + M_1$, $A_e(0) = fF + M_0$, M_1 is the set of all elements ec where c is in N and we have the corresponding result for M_0.

The following important known[3] lemma can now be stated.

LEMMA 1. *Let A be a commutative power-associative algebra of degree two over a field F of characteristic zero. Then $A_e(1/2)A_e(1) \subseteq A_e(1/2) + M_0$ and $A_e(1/2)A_e(0) \subseteq A_e(1/2) + M_1$. Note that the result of the lemma is not vacuous here since we are assuming u-stability only.*

Consider the product $(eB)G$ which was used to obtain (4) and (5). By Lemma 1, $(eB)G \subseteq A_e(1/2) + M_0$ so that $(b+wb)g = a(u-v) + h + c - wc$ for a, b in B, g, h in G, and c in N the radical of B. Then $(wb)g = a(u-v) + c$ and it is shown in [3] that $a = -d$ of relation (3). Also[4] the quantity d in (3) and (5) is in N. These results may be stated as follows.

LEMMA 2. *Let A be a u-stable commutative power-associative algebra over a field of characteristic zero. Then $GB \subseteq G + wN$, $G[B(u-v)] \subseteq wN$, $w(GB) \subseteq N$, $(wB)G \subseteq N'$, and $w(GB) + (wB)G \subseteq N(u-v)$.*

It will also be necessary to have

LEMMA 3. *The product $G\{(wB)[B(u-v)]\} \subseteq N$.*

For proof substitute $x = g$, $y = a$, $z = b(u-v)$ into the multilinear

[2] By Theorem 2 of [2].
[3] See Theorem 6 of [5].
[4] [2, Lemma 10].

identity obtained from the associativity of fourth powers.[5] Relation (1) implies $wz=w(az)=0$ and we have $wg=0$ by definition of G. Thus

$$4(wa)(gz) = w[(ga)z + (gz)a + g(az)] + g[(wa)z] + a[(gz)w] + z[(wa)g + w(ga)].$$

By (3) and (2), $(wa)(gz)$ is in $(wB)(wN)\subseteq N$. The quantity ga is in $G+wN$ by (4); hence $(ga)z$ is in $G[B(u-v)]+(wN)[B(u-v)]$. Consequently, (3) and (6) imply $w[(ga)z]$ in N. Since $(gz)a$ lies in $\{G[B(u-v)]\}B\subseteq(wN)B\subseteq wN$, $w[(gz)a]$ is in N. Also $w[g(az)]$ is in $w\cdot G[B(u-v)]\subseteq w(wN)=N$. The product $a[(gz)w]$ is in N and $z[(wa)g+w(gz)]$ is contained in $[B(u-v)]\cdot[N(u-v)]\subseteq N$. This completes the proof of Lemma 3.

The proofs of Lemmas 15 and 17 of [**3**] which state that $[A_u(1/2)\cdot N']C\subseteq N'A_u(1/2)$ and $[A_u(1/2)N']A_u(1/2)\subseteq N'$ follow without change. We also have without change that $N'+A_u(1/2)N'$ is an ideal of A. Since A is simple, this ideal must be zero because it does not contain the identity element. Thus $A=uF+vF+A_u(1/2)$, which is a Jordan algebra. A Jordan algebra is stable so we have as a corollary that a simple commutative power-associative algebra of degree 2 and characteristic 0 is stable if and only if it is u-stable.

Bibliography

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[5] The identity is stated in all of our references.