FREE IDEALS IN RINGS OF FUNCTIONS

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1. **Introduction.** Let X be any infinite set, and R a ring with unit element e. Let $A(X, R)$ be the ring of all functions from X to R, with the usual definitions of $+$ and $\cdot$. No topological considerations are introduced; i.e., all the sets involved are taken as discrete. Let I be an ideal in A. Following Hewitt [2] and Kaplansky [3], we say that I is *free* if and only if for each $x \in X$ there exists $f \in I$ such that $f(x) = e$. The purpose of this paper is to give an exact characterization of all the free left ideals of A. The results take a particularly simple form if we make the additional assumption that every left ideal in R is principal.

2. **Preliminary definitions and lemmas.** Let us denote by L the set of all left ideals of R. We shall consider L as a lattice under the usual operations of $+$ and $\cap$. We admit $\{0\}$ and R as elements of L.

We denote by L^X the set of all functions from X to L. If $p \in L^X$, $q \in L^X$, we define $p = q$ to mean that $p(x) = q(x)$ for all but a finite number of $x \in X$. We define $p + q$ by $(p+q)(x) = p(x) + q(x)$, and $p \cap q$ by $(p \cap q)(x) = p(x) \cap q(x)$. Under these operations L^X becomes a lattice, in which $p < q$ means that $p(x) \subset q(x)$ for all but a finite number of $x \in X$. The function in L^X which is identically 0 will be denoted by θ. For each $p \in L^X$ we define $\mu(p) = \{x \in X \mid p(x) \neq R\}$.

The set of all subsets of X will be denoted by 2^X. If $\alpha \in 2^X$, $\beta \in 2^X$, we define $\alpha = \beta$ to mean that α and β are identical save for a finite set of points. We denote the empty set by $\varnothing$. Thus $\alpha = \varnothing$ means that α is a finite subset of X. We consider 2^X as a lattice under the usual operations of $\cup$ and $\cap$. In this lattice $\alpha \subset \beta$ means that all but a finite number of points of α lie in β.

The set of all free left ideals of $A(X, R)$ will be denoted by $F(A)$.

The proof of the following lemma may be left to the reader.

LEMMA 1. *Let $x_1, x_2, \cdots, x_n$ be a finite number of points of X, and let $a_1, a_2, \cdots, a_n$ be arbitrary elements of R. Then for any $I \in F(A)$ there exists $f \in I$ such that $f(x_i) = a_i$ for $i = 1, 2, \cdots, n$, and $f(x) = 0$ for all other $x \in X$.*

Let us write

$$J_0 = \{f \in A \mid f(x) = 0 \text{ for all but a finite number of } x \in X\}.$$

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Clearly $J_0 \in F(A)$. If I is any left ideal in A, Lemma 1 implies that $I \in F(A)$ if and only if $I \supset J_0$. Thus the intersection of any number of ideals in $F(A)$ is again an ideal in $F(A)$. Also it is obvious that $I_1 \in F(A)$ and $I_2 \in F(A)$ imply $I_1 + I_2 \in F(A)$. Hence $F(A)$ is a lattice with respect to $+$ and $\cap$, and the ideal J_0 is its 0 element.

For $\alpha \in 2^X$, $p \in L^X$, we now define

$$J(\alpha, p) = \{f \in A \mid f(x) \in p(x) \text{ for all but a finite number of } x \in \alpha\},$$

with the agreement that $J(\emptyset, p) = A$ for all p. It is clear that $J(\alpha, p) \in F(A)$, and that $J(X, \theta) = J_0$. The following facts are also obvious.

LEMMA 2.
(1) *$J(\alpha, p) = A$ if and only if $\mu(p) \cap \alpha = \emptyset$.*
(2) *$J(\alpha, p) \subset J(\beta, p)$ if and only if $\beta \subset \alpha$.*
(3) *$J(\alpha, p) \subset J(\alpha, q)$ if and only if $p(x) \subset q(x)$ for all but a finite number of $x \in \alpha$.*

LEMMA 3. $J(\alpha, p) + J(\beta, q) = J(\alpha \cap \beta, p + q)$.

PROOF. Suppose $f \in J(\alpha, p)$, $g \in J(\beta, q)$. Then for all but a finite number of $x \in \alpha \cap \beta$, we have $f(x) + g(x) \in p(x) + q(x)$. Hence $J(\alpha, p) + J(\beta, q) \subset J(\alpha \cap \beta, p + q)$. Conversely, suppose that $f \in J(\alpha \cap \beta, p + q)$. We define functions f_1 and f_2 in A as follows:

$f_1 = 0$ on the complement of $\alpha \cup \beta$,
$f_1 = f$ on $\beta \cap \alpha'$,
$f_1 = 0$ on $\alpha \cap \beta'$,
$f_2 = f$ on the complement of $\alpha \cup \beta$,
$f_2 = 0$ on $\beta \cap \alpha'$,
$f_2 = f$ on $\alpha \cap \beta'$.

For $x \in \alpha \cap \beta$ we have $f(x) = a_x + b_x$, where $a_x \in p(x)$ and $b_x \in q(x)$ for all but a finite number of $x \in \alpha \cap \beta$. Thus for $x \in \alpha \cap \beta$, we define $f_1(x) = a_x$, $f_2(x) = b_x$. Then $f_1 \in J(\alpha, p)$, $f_2 \in J(\beta, q)$, and $f = f_1 + f_2$. Hence $J(\alpha \cap \beta, p + q) \subset J(\alpha, p) + J(\beta, q)$.

Now for $f \in A$, let us write

$$\sigma(f) = \{x \in X \mid f(x) \text{ has no left inverse}\},$$
$$\lambda(f) = \text{complement of } \sigma(f).$$

Also, for $f \in A$, we define $p_f \in L^X$ by

$$p_f(x) = [f(x)] = \text{left ideal generated by } f(x).$$

LEMMA 4. *If $I \in F(A)$ and $f \in I$, then $J(\sigma(f), p_f) \subset I$.*

PROOF. Suppose $g \in J(\sigma(f), p_f)$. Then $g(x) \in p_f(x)$ for all $x \in \sigma(f)$

except for a finite set of points $x_1, x_2, \cdots, x_n$. I.e., for each $x\in\sigma(f)$, $x\neq x_i$, there exists $a_x\in R$ such that $g(x)=a_xf(x)$. For $x\in\lambda(f)$, let $f^{-1}(x)$ denote any left inverse of $f(x)$. Let us define a function g_1 in A as follows:

$$g_1(x)=a_x \text{ for } x\in\sigma(f)-\{x_1, x_1, \cdots, x_n\},$$

$$g_1(x)=0 \text{ for } x=x_i,\ i=1, 2, \cdots, n,$$

$$g_1(x)=g(x)\cdot f^{-1}(x), \text{ for } x\in\lambda(f).$$

Then for $i=1, 2, \cdots, n$ we have $(g_1f)(x_i)=0$, and $(g_1f)(x)=g(x)$ for all other $x\in X$. But by Lemma 1, there exists $h\in I$ such that $h(x_i)=g(x_i)$ for $i=1, 2, \cdots, n$, and $h(x)=0$ for all other x. Then $g=g_1f+h\in I$.

3. **Structure of the free left ideals of** $A(X, R)$. Following Birkhoff [1, p. 21], we introduce the following definitions.

DEFINITION. A subset K of L^X is an *ideal* in L^X if and only if

(1) $p\in K$ and $q\in K$ imply $p+q\in K$,

(2) $p\in K$ and $q<p$ imply $q\in K$.

The set of all ideals of L^X will be denoted by $\mathcal{K}(X, R)$. We admit $\{\theta\}$ and L^X as elements of $\mathcal{K}(X, R)$.

DEFINITION. A subset D of 2^X is a *dual ideal* in 2^X if and only if

(1) $\alpha\in D$ and $\beta\in D$ imply $\alpha\cap\beta\in D$,

(2) $\alpha\in D$ and $\beta\supset\alpha$ imply $\beta\in D$.

The set of all dual ideals of 2^X will be denoted by $\mathcal{D}(X)$. We admit $\{X\}$ and 2^X as elements of $\mathcal{D}(X)$.

Now for $D\in\mathcal{D}(X)$ and $K\in\mathcal{K}(X, R)$ we define

$$J(D, K) = \bigcup_{\alpha\in D, p\in K} J(\alpha, p),$$

where the "$\cup$" denotes the set-theoretic union of the $J(\alpha, p)$. We verify that $J(D, K)\in F(A)$. Suppose that f and g are functions in $J(D, K)$. Then there exist α and β in D, and p and q in K, such that $f\in J(\alpha, p)$ and $g\in J(\beta, q)$. By Lemma 3, $f+g\in J(\alpha\cap\beta, p+q)\subset J(D, K)$, from which it follows that $J(D, K)$ is a left ideal.

Also note that $J(D, K)=A$ implies that the function which is identically equal to e is in $J(\alpha, p)$ for some $\alpha\in D$ and $p\in K$; from this it follows that $J(D, K)=A$ if and only if $J(\alpha, p)=A$ for some $\alpha\in D$, $p\in K$.

LEMMA 5. *$I\in F(A)$ implies $I=\bigcup_{\alpha\in D, g\in I}J(\alpha, p_g)$ for some $D\in\mathcal{D}(X)$.*

PROOF. Define $D=\{\sigma(f)\mid f\in I\}$. (It is obvious that $D=2^X$ if and only if $I=A$.) We show that $D\in\mathcal{D}(X)$. Clearly, $\alpha\in D$ and $\gamma\supset\alpha$ imply

$\gamma\in D$. (Use the function which is 0 on γ and e on $X\cap\gamma'$.) Now suppose $\alpha\in D$, $\beta\in D$, $\alpha=\sigma(f)$, $\beta=\sigma(g)$, where f, $g\in I$. Let us define $f^*\in A$ and $g^*\in A$ as follows:

$$\begin{aligned}
&f^*(x)=\text{any left inverse of } f(x), \text{ for } x\in\lambda(f),\\
&f^*(x)=0 \text{ for } x\in\sigma(f),\\
&g^*(x)=0 \text{ for } x\in\sigma(g),\\
&g^*(x)=\text{any left inverse of } g(x), \text{ for } x\in\lambda(g)\cap\sigma(f),\\
&g^*(x)=0 \text{ for } x\in\lambda(g)\cap\lambda(f).
\end{aligned}$$

Then $(f^*f+g^*g)(x)=0$ for $x\in\sigma(f)\cap\sigma(g)$, and $(f^*f+g^*g)(x)=e$ for all other $x\in X$. Hence $\sigma(f^*f+g^*g)=\sigma(f)\cap\sigma(g)=\alpha\cap\beta$. But $f^*f+g^*g\in I$. Hence $\alpha\cap\beta\in D$, and it follows that $D\in\mathfrak{D}(X)$.

Now let f and g be arbitrary functions in I. By Lemma 4, $J(\sigma(f), p_f)\subset I$, $J(\sigma(g), p_g)\subset I$; and hence by Lemma 3, $J(\sigma(f)\cap\sigma(g), p_f+p_g)\subset I$. Using (2) and (3) of Lemma 2, we then have

$$J(\sigma(f), p_g)\subset J(\sigma(f)\cap\sigma(g), p_g)\subset J(\sigma(f)\cap\sigma(g), p_f+p_g)\subset I,$$

and hence $\bigcup_{\alpha\in D, g\in I} J(\alpha, p_g)\subset I$. Since it is obvious that we also have $I\subset\bigcup_{\alpha\in D, g\in I} J(\alpha, p_g)$, the lemma is proved.

We are now ready for our main result.

THEOREM 1. *Let R be a ring with unit in which each left ideal is principal, and let A be the ring of all functions from X to R. Then $I\in F(A)$ if and only if $I=J(D, K)$ for some $D\in\mathfrak{D}(X)$ and $K\in\mathfrak{K}(X, R)$.*

PROOF. Define D as in Lemma 5. Let $K=\{p_g \mid g\in I\}$. We show that $K\in\mathfrak{K}(X, R)$. First suppose that $p_g\in K$ and $q<p_g$. Then $q(x)\subset p_g(x)$ for all $x\in X$ save for a finite set of points $x_1, x_2, \cdots, x_n$. Since $g(x)$ generates the ideal $p_g(x)$, then for $x\neq x_i$, $i=1, 2, \cdots, n$, there exists $m_x\in R$ such that $m_xg(x)$ generates the ideal $q(x)$. Define a function $f_1\in A$ by $f_1(x)=m_x$ for $x\neq x_i$, and $f_1(x_i)=0$ for $i=1, 2, \cdots, n$. Let a_i be an element of R which generates the ideal $q(x_i)$. By Lemma 1, there exists $f_2\in I$ such that $f_2(x_i)=a_i$ for $i=1, 2, \cdots, n$, and $f_2(x)=0$ for all other $x\in X$. Then $f=f_1g+f_2\in I$, and $p_f=q$. Hence $q\in K$.

Now suppose that $g\in I$ and $h\in I$. For each $x\in X$, let c_x be an element of R which generates the ideal $p_g(x)+p_h(x)$. Let f be the function in A such that $f(x)=c_x$ for all x. Then $p_f=p_g+p_h$. But $f\in J(\sigma(g)\cap\sigma(h), p_g+p_h)\subset I$. Hence $p_g+p_h\in K$, and $K\in\mathfrak{K}(X, R)$. The theorem now follows from Lemma 5.

4. **A special case.** In the special case when R is a division ring, the above discussion is of course greatly simplified; and we can also easily obtain an abstract characterization of the lattice $F(A)$. Assuming now that R is a division ring, we define, for $\alpha\in 2^X$,

$$J(\alpha) = \{f \in A \mid f(x) = 0 \text{ for all but a finite number of } x \in \alpha\},$$

with the agreement that $J(\emptyset) = A$. The following relations are easily verified.

LEMMA 6. *For $\alpha \in 2^X$, $\beta \in 2^X$, we have*

$$J(\alpha) + J(\beta) = J(\alpha \cap \beta),$$
$$J(\alpha) \cap J(\beta) = J(\alpha \cup \beta).$$

For $D \in \mathcal{D}(X)$, we now define $J(D) = \bigcup_{\alpha \in D} J(\alpha)$. We then obtain the following form of Theorem 1, making the appropriate simplifications in the proof. This result has, in essence, already been obtained by Hewitt [2, Theorem 36]. We omit the details.

THEOREM 1′. *Let R be a division ring, A the ring of all functions from X to R. Then $I \in F(A)$ if and only if $I = J(D)$ for some $D \in \mathcal{D}(X)$.*

In this special case we now show how to construct from the set X a lattice-isomorphic image of the lattice $F(A)$. First we prove

LEMMA 7. *$J(D_1) = J(D_2)$ if and only if $D_1 = D_2$.*

PROOF. Suppose $J(D_1) = J(D_2)$, and $\alpha \in D_1$. The function f which is 0 on α and e on the complement of α is in $J(\alpha) \subset J(D_1)$. Then $f \in J(\beta)$ for some $\beta \in D_2$. This means $\alpha \supset \beta$, whence $\alpha \in D_2$. Hence $D_1 \subset D_2$, and likewise $D_2 \subset D_1$.

Now for $D_1 \in \mathcal{D}(X)$, $D_2 \in \mathcal{D}(X)$, we define

$$D_1 \cap D_2 = \text{set-theoretic intersection of } D_1 \text{ and } D_2,$$

$$D_1 + D_2 = \{\gamma \in 2^X \mid \gamma \supset \alpha \cap \beta \text{ for some } \alpha \in D_1 \text{ and } \beta \in D_2\}.$$

It is easily verified that $\mathcal{D}(X)$ forms a lattice under these operations. We then obtain the following theorem, the proof of which will be left to the reader.

THEOREM 2. *Let R be a division ring, and A the ring of all functions from X to R. Then the correspondence $J(D) \leftrightarrow D$ is a lattice-isomorphism of $F(A)$ with $\mathcal{D}(X)$.*

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