GENERATORS OF THE RING OF BOUNDED OPERATORS

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J. A. Dieudonné suggested in conversation that some small number of projections might suffice to generate the ring $\mathcal{B}$ of all bounded operator on *separable* Hilbert space $\mathcal{H}$. There is some analogy between such a result and the theorem that a compact connected metric group can be generated by two elements.[1] The analogy is still closer for Theorem 2 below.

In this paper, operators $A, \cdots$ will be said to "generate" the smallest ring (i.e., *weakly* closed self-adjoint algebra) containing $A, \cdots$ and the constants.

THEOREM 1. *There exist three projections which generate the ring $\mathcal{B}$. The number three cannot be reduced if $\mathcal{H}$ has dimensionality 3 or greater.*

If $\dim(\mathcal{H})=1$, there is nothing but constants in $\mathcal{B}$. If $\dim(\mathcal{H})=2$, then any two noncommuting projections generate $\mathcal{B}$. Hereafter suppose $\dim(\mathcal{H})\geqq 3$.

I will use what I shall call the *closeness operator* $C=C(E, F)$ associated with any two projections E, F. It is defined by $C(E, F)=1-E-F+EF+FE=EFE+(1-E)(1-F)(1-E)$. It is a positive definite operator which, in case[2] $E\cap F+E\cap(1-F)+(1-E)\cap F+(1-E)\cap(1-F)=0$, acts like "the square of the cosine of the angle" between $E\mathcal{H}$ and $F\mathcal{H}$.

To show E and F fail to generate $\mathcal{B}$, I shall show some nonconstant operator commutes with both; this is enough because such an operator commutes with the whole ring generated by E and F, whereas the commutator of $\mathcal{B}$ contains only constants. Since $C(E, F)$ commutes with E and F, the only case to be considered is C constant. $E\neq 0$ may be assumed. Choose $x=Ex$, $x\neq 0$, x otherwise arbitrary. Now the subspace spanned by x and Fx is not zero, and since it is at most 2-dimensional it is not $\mathcal{H}$; so the projection P on it is nonconstant. $P\mathcal{H}$ is invariant under E and F, since $EFx=EFEx=Cx$, which in this case is a multiple of x. Therefore P is a nonconstant operator

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[1] H. Auerbach, *Sur les groupes linéaires bornées* (III), Studia Mathematica vol. 5 (1934) pp. 43–49. J. Schreier and S. Ulam, *Sur le nombre des générateurs d'un groupe topologique compact et connexe*, Fund. Math. vol. 24 (1935) pp. 302–304.

[2] Here "$\cap$" is intersection in the lattice of projections. This equation says $E\mathcal{H}$ and $F\mathcal{H}$ are in position p (J. Dixmier, *Position relative de deux variétés linéaires fermées dans un espace de Hilbert*, Rev. Sci. vol. 86 (1948) pp. 387–399).

commuting with E and F.

Now for the proof of the first sentence in the theorem. Only countable dimensionality will be treated, finite dimensionality is handled similarly. There is a good deal of leeway in the construction; the particular generators E_1, E_2, E_3 given here are chosen for convenience.

Let $x_1, y_1, x_2, y_2, x_3, \cdots$ be an orthonormal basis of $\mathfrak{H}$. Let $z_n = \cos\theta_n x_n + \sin\theta_n y_n$, $n=1, 2, \cdots$, with $\theta_n = \pi/(2n+1)$. Let P_n be the projection on $[x_n, y_n]$, the subspace spanned by x_n and y_n, $n=1, 2, \cdots$. Let E_1 be the projection on $[x_1, x_2, x_3, \cdots]$; E_2, the projection on $[z_1, z_2, z_3, \cdots]$.

Now the ring generated by E_1 and E_2 contains $C=C(E_1, E_2)$. It can be shown by a direct computation that the eigenspaces of C are the $P_n\mathfrak{H}$, the corresponding eigenvalues being $\cos^2\theta_n$. Each eigenvalue is an isolated point of the spectrum; the characteristic function of the set containing only $\cos^2\theta_n$ is measurable (even continuous) on the spectrum of C. The spectral theorem implies that the ring contains all the P_n. Also the ring contains every operator on $P_n\mathfrak{H}$, for on that 2-dimensional Hilbert space E_1 and E_2 are noncommuting projections (see the remark at the beginning of the proof).

Finally, define E_3 as the projection on

$$[x_1 + x_2,\ y_2 + y_3,\ \cdots,\ x_{2n-1} + x_{2n},\ y_{2n} + y_{2n+1},\ \cdots].$$

The ring $\mathfrak{R}$ generated by E_1, E_2, and E_3 will be shown to be $\mathfrak{B}$.

Let $E(x; y)$ denote, for any unit vectors x and y, the operator characterized by

$$E(x; y)x = y,$$
$$z \perp x \quad \text{implies} \quad E(x; y)z = 0.$$

Also hereafter let w_n mean either x_n or y_n, $n=1, 2, \cdots$.

$\mathfrak{R}$ contains $E(x_{2n-1}; x_{2n})$. For it contains the projection on $[x_{2n-1}]$, and premultiplying that projection by $2P_{2n}E_3$ gives the desired operator. Therefore $\mathfrak{R}$ contains $E(w_{2n-1}; w_{2n})$, and necessarily also its adjoint, $E(w_{2n}; w_{2n-1})$. Similarly, $\mathfrak{R}$ contains $E(y_{2n}; y_{2n+1})$, hence $E(w_{2n}; w_{2n+1})$ and $E(w_{2n+1}; w_{2n})$. By induction, $E(w_i; w_j)\in\mathfrak{R}$. Therefore $\mathfrak{R}$ contains every operator whose matrix, using the originally given orthonormal basis of $\mathfrak{H}$, has finitely many nonzero entries.

Let $A\in\mathfrak{B}$, and let

$$A_n = \sum_{1}^{n} P_k A P_k.$$

Then $A_n \in \mathfrak{R}$, and A is the weak limit of the A_n, so $A \in \mathfrak{R}$. It has been proved that $\mathfrak{R} = \mathfrak{B}$.

(Weak closure of $\mathfrak{R}$ was required only in the last paragraph of the proof; uniform closure was all that was used before. I do not know if there exist three projections which generate $\mathfrak{B}$ by algebraic operations and *uniform* limits.[3])

THEOREM 2. *There exist two unitary operators which generate* $\mathfrak{B}$. *They may be chosen so one of them is a symmetry.*

This will be proved relying largely on the previous proof, and keeping the same symbols. Again I shall treat only the countable case.

Let $Ux_n = z_n$, $Uy_n = -\sin\theta_n x_n + \cos\theta_n y_n$. This defines U as a unitary operator. Let $V = 1 - 2E_3$, a symmetry. The ring $\mathfrak{R}'$ generated by U and V will be shown to be $\mathfrak{B}$.

$E_3 = (1 - V)/2 \in \mathfrak{R}'$. As above, one shows first that each $P_n\mathfrak{H}$ is an eigenspace of $U + U^*$, corresponding to the eigenvalue $2\cos\theta_n$; and therefore that $P_n \in \mathfrak{R}'$. The last part of the previous proof can be invoked once it is shown that $\mathfrak{R}'$ contains every operator on the 2-dimensional subspace $P_n\mathfrak{H}$. But $2P_nE_3P_n \in \mathfrak{R}'$ and $2UP_nE_3P_nU^* \in \mathfrak{R}'$ are noncommuting projections operating on $P_n\mathfrak{H}$.

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[3] The projections given here do not. In fact, every operator A which is a uniform limit of polynomials in E_1, E_2, and E_3, has the special property (among others) that (Ax_{2n}, x_{2n}) has a limit as $n \to \infty$. Proof omitted.