ON THE MAXIMAL DILATION OF QUASICONFORMAL MAPPINGS

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1. Let G and G' be two plane open sets and $w(z)$ a topological mapping of G onto G'. By Q we denote any quadrilateral in G, i.e. the topological image of a closed square with a distinguished pair of opposite sides. The conformal modulus m of Q is the ratio $m=a/b$ of the sides of a conformally equivalent rectangle R, the distinguished sides of Q corresponding to the sides of length b. We call this essentially unique conformal mapping of Q onto R the canonical mapping of Q. The modulus m is equal to the extremal distance of the two distinguished sides of Q with respect to Q. The *maximal dilation* of the mapping $w(z)$ on G is the number

$$K[w(z)] = K = \sup_{Q} \frac{m'}{m}$$

where m' denotes the modulus of the image Q' of Q under the mapping $w(z)$ and Q varies over all possible quadrilaterals. The mapping is said to be quasiconformal if K is finite.

Given a closed subset $E \subset G$ (closed only with respect to G). Then, $G-E$ is open, and if we denote by $K_0[w(z)]=K_0$ the maximal dilation of $w(z)$ on $G-E$, we get $K_0 \leqq K$. We are looking for sufficient conditions on E such that $K_0=K$. The answer will be different if we consider only the class of all quasi-conformal mappings of G or the larger class of all topological mappings, including the ones with infinite maximal dilation. We call a point set E which allows the conclusion $K_0=K$ deletable for the class in consideration. It was proved by Ahlfors in [1] that analytic arcs are deletable for the class of all topological mappings.[1] It is also proved there that $K=\sup_{\tilde{Q}} m'/m$, where $\tilde{Q}$ denotes any analytic quadrilateral, i.e. a quadrilateral Q with a canonical mapping which is conformal in an open neighborhood of Q, and furthermore that K does not become larger if the boundary curves of the quadrilaterals are allowed to have points in common with the boundary of G.

2. If E is a discrete point set, we can consider the slightly more general problem that $w_0(z)$ is only known to be quasiconformal with maximal dilation K_0 on $G-E$, without knowing that $w_0(z)$ is a topological mapping of the whole open set G. What are the conditions on

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[1] Theorem 4, p. 9.

E in order that for every $w_0(z)$ there is a topological mapping $w(z)$ of G which coincides with $w_0(z)$ on $G-E$ and has the same maximal dilation. We call $w(z)$ the continuation of $w_0(z)$. It is clear that there can exist only one topological continuation.

THEOREM 1. *A necessary and sufficient condition that every quasiconformal mapping $w_0(z)$ of $G-E$ has a continuation to G with the same maximal dilation is that every compact subset of E is a nullset O_{AD}.*[2]

The condition is necessary, for, if E_0 is any compact subset of E which is not a set O_{AD}, there exists a parallel slit mapping of the complement of E_0, which is conformal outside E_0 and cannot be extended conformally over E_0.

If, on the other hand, E possesses the property of the theorem, then to every point $z_0 \subset E$ there exists a sequence of nonoverlapping doubly connected ring-domains in $G-E$ with a divergent sum of moduli $\mu_i(\mu_i =$ extremal distance of the two boundary components of a ring domain), a property which is invariant under the quasiconformal mapping of $G-E$. But a boundary component with this property must necessarily be a point. Therefore every point of E goes over into a point, and it is readily seen that $w_0(z)$ has a topological continuation $w(z)$ over E. It follows from the succeeding lemma that the maximal dilation of $w(z)$ on G is K_0.

LEMMA 1. *If Q is an analytic quadrilateral in G and E_0 a compact O_{AD} set in Q, then to every $\epsilon>0$ there exists a finite system of simply connected, nonoverlapping longitudinal*[3] *strips S_i, $S_i \subset Q-E_0$ with modulus m_i, such that $\sum_i 1/m_i \geqq 1/m-\epsilon$.*

From that we get, because of $m_i' \leqq K_0 m_i$,

$$\frac{1}{m'} \geqq \sum_i \frac{1}{m_i'} \geqq K_0^{-1} \sum_i \frac{1}{m_i} \geqq K_0^{-1}\left(\frac{1}{m} - \epsilon\right)$$

and therefore

$$m' \leqq K_0 m$$

for each analytic quadrilateral in G.

The lemma can be proved in the same way as Theorem 9 in Ahlfors and Beurling [2]. We map Q onto a rectangle R with sides a and b by means of its canonical conformal mapping. E_0 is transformed

[2] I.e. a set which allows no nonconstant and single-valued analytic function with a bounded Dirichlet-integral in its complement.

[3] That is to say the boundary of S_i has an interval in common with each distinguished side of Q; S_i is therefore a quadrilateral with these intervals as distinguished sides.

into a set O_{AD} which we denote by E_0'. To any given $\epsilon>0$ we can find a concentric rectangle R' with sides $a'>a$ and $b'<b$ and such that $b'/a'\geqq b/a-\epsilon/2$. A curvilinear rectangle R'' which is contained in the rectangle with sides a', b and contains the rectangle with sides a, b' and the sides of which do not meet E_0' can be constructed; we choose its distinguished sides outside R and such that its modulus is $\leqq a'/b'$. The set $E_0'\cap R''$ is a compact subset of the open rectangle R'', and by an exhaustion we can obviously find the strips S_i'' in R'' with $\sum_i 1/m_i''\geqq 1/m''-\epsilon/2$. Each S_i'' contains a longitudinal strip S_i of the original rectangle R with modulus $m_i\leqq m_i''$. Therefore we get

$$\sum_i \frac{1}{m_i} \geqq \sum_i \frac{1}{m_i''} \geqq \frac{1}{m''} - \epsilon/2 \geqq b'/a' - \epsilon/2 \geqq b/a - \epsilon = \frac{1}{m} - \epsilon.$$

3. For the composition of piecewise quasiconformal mappings however the stress lies on connected, not on discrete point sets. To get an answer in this direction, we consider a rectangle R in the z-plane $(0\leqq x\leqq a,\ 0\leqq y\leqq b)$ and a topological mapping $w(z)$ of R onto a rectangle R' in the w-plane $(0\leqq u\leqq a',\ 0\leqq v\leqq b')$ which preserves the four sides respectively. By E_y we denote the intersection of the line $\Im z=y$ with the given closed set E in R, and by $L_u(y)$ the linear measure of the vertical projection (i.e. onto the u-axis) of the w-image E_y' of E_y. The modulus of a horizontal rectangle is its length divided by its height; the modulus of its w-image has to be taken with respect to the sides which correspond to the vertical sides of the rectangle.

LEMMA 2. *If* (1) *the modulus m of every horizontal rectangle in R which has no interior point in common with E and the modulus m' of its image satisfy $m'\leqq Km$, where K is some positive constant, and*

(2) *the linear measure $L_u(y)$ of the vertical projection of E_y' is zero for almost every y, we have*

$$a'/b' \leqq Ka/b.$$

PROOF. For any $\epsilon>0$ the set O_ϵ of all values y with $L_u(y)<\epsilon$ is open in $0\leqq y\leqq b$ and has linear measure b. For any $y\in O_\epsilon$ we can find an interval $y_1<y<y_2$ and a family of finitely many rectangles R_i with sides on $\Im z=y_1$ and $\Im z=y_2$ which contain every E_y for $y_1<y<y_2$ and such that their w-images have a vertical projection of total linear measure less than ϵ. Let now B be a closed subset of O_ϵ of measure $>b-\epsilon$. We have an open covering of B by means of intervals β of the above kind, and by the Heine-Borel theorem there exists a finite covering, say $\beta_1, \cdots, \beta_n$. Starting with β_1 and the rectangles in the corresponding horizontal strip $(0\leqq x\leqq a,\ y\in\beta_1)$, we take in every new interval β_i only that part which does not lie in one of the former intervals, and restrict the corresponding horizontal strip and its rectangles in the same way. We get a system of finitely many strips S_i of height b_i with total vertical measure $\sum_i b_i \geqq b-\epsilon$. The rectangles T_{ij} in S_i, complementary to the rectangles R_{ij} in S_i, have no interior point in common with E and thus their moduli satisfy $m'_{ij} \leqq K m_{ij}$. As the vertical projection of the images R'_{ij} of the R_{ij} has a total linear measure $<\epsilon$, there exist finitely many closed, disjoint intervals on the u-axis of total measure $<\epsilon$ covering this projection. We denote the complementary intervals on the u-axis by α'_{ij}, their length by a'_{ij}: we have $\sum_j a'_{ij} > a' - \epsilon$. Each α'_{ij} is spanned by the image T'_{ij} of at least one T_{ij}, and the modulus of this T'_{ij} is therefore $\geqq a'^2_{ij}/A'_{ij}$, A'_{ij} denoting the area of (the interior of) T'_{ij}. From that we get for each strip S_i

$$\frac{a}{b_i} \geqq \sum_j m_{ij} \geqq \frac{1}{K}\sum_j m'_{ij} \geqq \frac{1}{K}\sum_j \frac{a'^2_{ij}}{A'_{ij}}$$
$$\geqq \frac{1}{K}\Big(\sum_j a'_{ij}\Big)^2 \Big/ \sum_j A'_{ij} \geqq \frac{1}{K}\,\frac{(a'-\epsilon)^2}{A'_i}$$

with A'_i = area of S'_i. Taking the reciprocals and summing up we get

$$\frac{b-\epsilon}{a} \leqq \frac{\sum_i b_i}{a} \leqq K\frac{\sum_i A'_i}{(a'-\epsilon)^2} \leqq K\frac{a'b'}{(a'-\epsilon)^2}$$

and therefore

$$b/a \leqq K(b'/a')$$

which proves the lemma.

Let E be an arbitrary set of finite linear measure L in R. Then it is readily proved that the set of values y where E_y consists of at least N points has linear measure $\leqq L/N$, and therefore the set of y where E_y consists of infinitely many points has measure zero. If E is of Σ-finite measure, i.e. the sum of denumerably many sets E^i of finite linear measure, the set of values y for which E_y consists of nondenumerably many points has linear measure zero. But for any y where E_y is denumerable, the image set is denumerable and so is its projection, therefore $L_u(y)=0$. As the property to be of Σ-finite linear measure is carried over by a conformal mapping of the closed quadrilateral, we get the

THEOREM 2. *If E is a closed subset of G of Σ-finite linear measure and $w(z)$ any topological mapping of G, its maximal dilation on G is equal* *to its maximal dilation on $G-E$, that is to say E is deletable for the class of all topological mappings of G.*

If $E=E^1+E^2$ is the sum of a closed set E^1 of Σ-finite linear measure and a closed set E^2, every compact subset of which is a nullset O_{AD}, then it is clear from the construction that E is deletable. We can first delete E^1-E^2, which is a closed set of $G-E^2$, and then E^2.

From Theorem 2 we get the following generalization of Theorem 13 in Ahlfors and Beurling [2]:

COROLLARY. *A closed, discrete point set E of the complex plane, which is of Σ-finite linear measure, is a nullset O_{SB}*[4] *if and only if it is a nullset O_{AD}.*

For, if it is a nullset O_{SB} every schlicht conformal mapping of the complement of E is continuous on E and has therefore a conformal continuation, i.e. is a linear transformation. But this is known to be a sufficient condition for a closed discrete pointset to be an O_{AD} set. The converse is obvious.

4. If the mapping $w(z)$ is not only known to be topological in G but quasiconformal, that means has finite maximal dilation, and if outside E the maximal dilation is K_0, the point sets E which allow us to conclude $K_0=K$ are much larger.

LEMMA 3. *Let $w(z)$ be a topological mapping of R onto R' as in Lemma 2. If the two following conditions are fulfilled:*

(1) *For every horizontal rectangle in R and a certain positive number K we have $m'\leqq Km$;*

(2) *E is of two-dimensional measure zero; then the linear measure $L(y)$ of E'_y is zero for almost every y.*

PROOF. If this were not the case, we could find a closed subset B of $0\leqq y\leqq b$ of positive linear measure h and a positive number l such that the linear measure of E_y would be zero while the linear measure of E'_y would be $L(y)\geqq l$ for every $y\in B$. This is so because $L(y)$ is a measurable function of y and the linear measure of E_y is zero for almost all y.

If $y\in B$ is arbitrary, we can cover E_y by finitely many open (relatively to $0\leqq x\leqq a$) intervals α_j of length a_j. The images have length a'_j. We then take an interval $y_1<y<y_2$ and consider the rectangles $R_j(x\in\alpha_j,\ y_1\leqq y\leqq y_2)$. We call them the rectangles corresponding to the intervals α_j in the horizontal strip $(0\leqq x\leqq a,\ y_1\leqq y\leqq y_2)$. Every R_j is mapped onto a quadrilateral R'_j and we denote by l_j the inf.

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[4] I.e. the complement allows no schlicht, bounded conformal mapping.

of the length of all curves joining the two distinguished sides in R_j', by F_j the area of the open R_j'. Because E_y is of measure zero and $\sum_j a_j' \geqq l$, it is clear that, given any $y \in B$, we can choose the α_j and afterwards $y_1 < y < y_2$ in such a way that

$$\sum_j a_j \leqq \epsilon \quad \text{and} \quad \sum l_j \geqq l - \epsilon.$$

The intervals $y_1 < y < y_2$ provide us with an open covering of B from which we get a finite covering. As in Lemma 2 the restriction to distinct strips S_i does not change the above two conditions, and we have for any strip S_i (with its intervals α_{ij} of length a_{ij} and the corresponding rectangles R_{ij} with moduli m_{ij}) the following estimates:

$$\frac{a_{ij}}{b_i} = m_{ij} \geqq \frac{1}{K} m_{ij}' \geqq \frac{1}{K} \frac{l_{ij}^2}{F_{ij}}.$$

Therefore

$$\frac{\epsilon}{b_i} \geqq \sum_j a_{ij}/b_i \geqq \frac{1}{K} \sum_j \frac{l_{ij}^2}{F_{ij}} \geqq \frac{1}{K}\Big(\sum_j l_{ij}\Big)^2 \Big/ \sum_j F_{ij} \geqq \frac{1}{K} \frac{(l-\epsilon)^2}{\sum_j F_{ij}}.$$

Taking reciprocals and summing up we get

$$\frac{h}{\epsilon} \leqq \sum_i b_i/\epsilon \leqq K \sum_{ij} F_{ij}/(l-\epsilon)^2 \leqq K \frac{a'b'}{(l-\epsilon)^2}.$$

As $\epsilon \to 0$ we conclude $h=0$, q.e.d.

With exactly the same method but less rough estimates we can get the following result:

Lemma 3'. *Let E be any closed set in R, B a closed subset of $0 \leqq y \leqq b$ of measure h and such that for $y \in B$ the linear measure of E_y is $\leqq l$ while the linear measure of E_y' is $L(y) \geqq l'$. Let F denote the area of the closed set $\bigcup_{y \in B} E_y'$. Then we have*

$$l/h \geqq K^{-1}(l'^2/F).$$

For $l=0$, $l'>0$ we get $h=0$, i.e. the above theorem. For $l'=\infty$ (we have to replace $l'-\epsilon$ by a number $<l'$ in the proof) and $l=a$ we get $h=0$.

Corollary. *The set of values y for which the image curve of the segment $\Im z = y$ is not rectifiable is of measure zero.*

THEOREM 3. *If E is a closed subset of G of two-dimensional measure zero and $w(z)$ any quasiconformal mapping of G, its maximal dilation on $G-E$ is equal to its maximal dilation on G. In other words, a closed. set of zero area is deletable with respect to all quasiconformal mappings of G.*

PROOF. We consider any analytic quadrilateral $Q \subset G$ and map it as well as its image Q' conformally onto the rectangles R and R' respectively. As the set E_1 in R which corresponds to the part $E \cap Q$ of the exceptional set E in G is closed and of zero area, the conditions of Lemma 3 are fulfilled. From Lemma 3 we conclude that the conditions of Lemma 2 with the constant K_0 are fulfilled. From Lemma 2 we get therefore

$$a'/b' \leqq K_0(a/b) \qquad \text{q.e.d.}$$

5. Another application of the same method leads to the following Lemma 3'': A topological mapping of R onto R' with the property (1) of Lemma 3 is absolutely continuous on almost every horizontal line.

The set of all y for which the length $L_\eta(\xi)$ of the image of the stretch $(0=x=\xi,\ y=\eta)$ is not absolutely continuous in ξ is measurable. If it were not of measure zero, there would exist a closed set B on the interval $0 \leqq y \leqq b$ of positive linear measure h and a number l such that for every y in B the corresponding horizontal stretch carries a system of intervals of total length $<\epsilon$ while their images have total length $\geqq l$. The rest of the proof is a repetition of the one given for Lemma 3. Lemma 3'' proves the theorem on absolute continuity of quasiconformal mappings which was announced in the Bull. Amer. Math. Soc. Abstract 61-3-421, by the same author.

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