A NOTE ON ESTIMATING DISTRIBUTION FUNCTIONS[1]

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1. Statement of the problem. Let A be a positive number and for each positive integer n let $f_n(y)$ be a continuous function on the closed interval $[-A, A]$. Let $F(y)$ be a distribution function on $[-A, A]$. For each $n=1, 2, \cdots$, define a_n by

$$a_n = \int_{-A}^{A} f_n(y)dF(y). \tag{1.1}$$

In this note we consider the problem of estimating the distribution function $F(y)$ in terms of the sequence of numbers $\{a_n\}$, and the sequence of functions $f_n(y)$. To this end we consider, for each positive integer n, a system of equations and inequalities. We construct a distribution function $F_n(y)$ in terms of any solution of this system, and show that $\lim_{n\to\infty} F_n(y)=F(y)$ for every continuity point of $F(y)$.

2. Conditions for uniqueness of F. It is clear that in order to be able to estimate F, we must assume that F is the unique distribution function satisfying (1.1). More precisely we shall make the following

ASSUMPTION. *Let $G(y)$ be any function of bounded variation defined on $[-A, A]$ and satisfying*

$$a_n = \int_{-A}^{A} f_n(y)dG(y), \qquad n = 1, 2, \cdots. \tag{2.1}$$

Then $F(y)-G(y)$ is identically constant.

In this section we shall derive a condition which is equivalent to the uniqueness assumption. To this end let B be the Banach space of continuous functions defined on $[-A, A]$ and normed by

$$\|f\| = \max_{y\in[-A,A]} |f(y)|. \tag{2.2}$$

Then we have

THEOREM 1. *A necessary and sufficient condition that F be unique is that the sequence $f_n(y)$ be fundamental in B.*

PROOF. Suppose that F is unique. Let B' be the closed linear manifold spanned by the sequence f_n, and suppose that B' is a proper

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subspace of B. Let $f_0 \in B - B'$, and let ϕ be a bounded linear functional defined on B with $\phi(f_0) = 1$, and $\phi(f) = 0$, for $f \in B'$. It is well known that such functionals exist. From the representation theorem for linear functionals on B it follows that there exists a function of bounded variation on $[-A, A]$, say $H(y)$, satisfying

$$\phi(f) = \int_{-A}^{A} f(y) dH(y), \qquad \text{for every } f \in B. \tag{2.3}$$

Now let $G(y) = F(y) + H(y)$. Clearly $H(y)$ is not identically constant, for $\int_{-A}^{A} f_0(y) dH(y) = 1$. On the other hand we have

$$a_n = \int_{-A}^{A} f_n(y) dG(y), \qquad n = 1, 2, \cdots, \tag{2.4}$$

since $\int_{-A}^{A} f_n(y) dH(y) = 0$ for every n. Since F is assumed to be unique, it follows that the sequence f_n is fundamental, thus proving necessity.

Conversely suppose that the sequence F_n is fundamental in B, and suppose that $G(y)$ is a function of bounded variation on $[-A, A]$ satisfying

$$\int_{-A}^{A} f_n(y) dF(y) = \int_{-A}^{A} f_n(y) dG(y), \qquad n = 1, 2, \cdots. \tag{2.5}$$

From the fact that strong convergence in B implies weak convergence in B, and from the fact that the sequence f_n is fundamental in B, it follows that equation (2.5) holds for every $f \in B$. Hence for every real number t we have

$$\int_{-A}^{A} e^{itv} dF(y) = \int_{-A}^{A} e^{itv} dG(y),$$

and the uniqueness of F follows from well-known properties of Fourier-Stieltjes transforms.

3. **Construction of the sequence** F_n. Let n be a given positive integer. Let $y_0 = -A, y_1, y_2, \cdots, y_n = A$ be a subdivision of $[-A, A]$ into n equal subintervals. For $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, define the numbers $M_{ij}^{(n)}$ and $m_{ij}^{(n)}$ by

$$M_{ij}^{(n)} = \max_{v_{j-1} \leqq v \leqq v_j} f_i(y), \qquad m_{ij}^{(n)} = \min_{v_{j-1} \leqq v \leqq v_j} f_i(y). \tag{3.1}$$

Consider the following system of equations and inequalities in the unknowns $H_1^{(n)}, \cdots, H_n^{(n)}$:

$$(3.2)\qquad \begin{aligned} &\text{(i)} \quad H_j^{(n)} \geqq 0, && j = 1, \cdots, n.\\ &\text{(ii)} \quad \sum_{j=1}^{n} H_j^{(n)} = 1. &&\\ &\text{(iii)} \quad \sum_{j=1}^{n} M_{ij}^{(n)} H_j^{(n)} \geqq a_i, && i = 1, \cdots, n.\\ &\text{(iv)} \quad \sum_{j=1}^{n} m_{ij}^{(n)} H_j^{(n)} \leqq a_i, && i = 1, \cdots, n. \end{aligned}$$

The system (3.2) clearly has the solution $H_j^{(n)} = \int_{y_{j-1}}^{y_j} dF$, $j = 1, \cdots, n$. Now let $H_1^{(n)}, \cdots, H_n^{(n)}$ be an arbitrary solution of (3.2). We define a distribution function $F_n(y)$ on $[-A, A]$ by

$$F_n(y) = \sum_{y_j \leqq y} H_j^{(n)}. \tag{3.3}$$

In the next section we shall prove

THEOREM 2. *For each point of continuity of $F(y)$ we have*

$$\lim_{n\to\infty} F_n(y) = F(y).$$

4. Proof of Theorem 2.

LEMMA 1. *Let r be a fixed positive integer. Then* $\lim_{n\to\infty} \int_{-A}^{A} f_r(y)dF_n(y) = a_r$.

PROOF. We have $\int_{-A}^{A} f_r(y)dF_n(y) = \sum_{j=1}^{n} f_r(y_j)H_j^{(n)}$ for every positive integer n. Hence

$$\sum_{j=1}^{n} m_{rj}^{(n)} H_j^{(n)} \leqq \int_{-A}^{A} f_r(y)dF_n(y) \leqq \sum_{j=1}^{n} M_{rj}^{(n)} H_j^{(n)},$$

and it is sufficient to show that

$$a_r = \lim_{n\to\infty} \sum_{j=1}^{n} m_{rj}^{(n)} H_j^{(n)} = \lim_{n\to\infty} \sum_{j=1}^{n} M_{rj}^{(n)} H_j^{(n)}.$$

Now for each $n \geqq r$, we have, in virtue of (3.2),

$$\sum_{j=1}^{n} m_{rj}^{(n)} H_j^{(n)} \leqq a_r \leqq \sum_{j=1}^{n} M_{rj}^{(n)} H_j^{(n)}.$$

Also

$$\sum_{j=1}^{n} [M_{rj}^{(n)} - m_{rj}^{(n)}] H_j^{(n)} \leqq \max_{j=1,\dots,n} [M_{rj}^{(n)} - m_{rj}^{(n)}].$$

Since $f_r(y)$ is uniformly continuous on $[-A, A]$, the desired result follows.

LEMMA 2. *For each* $f\in B$, *we have*

$$\lim_{n\to\infty}\int_{-A}^{A} f(y)dF_n(y) = \int_{-A}^{A} f(y)dF(y).$$

PROOF. Let $f\in B$, and let ϵ be a positive number. The sequence f_n is fundamental in B, and so we may choose a finite subset, say $f_{i_1}, \cdots, f_{i_r}$, and real numbers $c_1, \cdots, c_r$ with the property that

$$\left\| f - \sum_{j=1}^{r} c_j f_{ij} \right\| < \epsilon/3.$$

Without loss of generality we may assume that $\sum_{i=1}^{r} |c_j| > 0$. Now, from Lemma 1, we may choose an integer N, so that for $n \geqq N$ we have

$$\max_{j=1,\cdots,n}\left| \int_{-A}^{A} f_{i_j}(y)dF_n(y) - \int_{-A}^{A} f_{i_j}(y)dF(y)\right| < \epsilon \Big/ 3\sum_{j=1}^{r} |c_j|$$

and consequently

$$\left| \int_{-A}^{A}\left(\sum_{j=1}^{r} c_j f_{i_j}(y)\right)dF_n(y) - \int_{-A}^{A}\left(\sum_{j=1}^{r} c_j f_{i_j}(y)\right)dF(y) < \frac{\epsilon}{3}\cdot\right.$$

Then for $n \geqq N$, we have

$$\begin{aligned}
&\left| \int_{-A}^{A} f(y)dF_n(y) - \int_{-A}^{A} f(y)dF(y)\right| \\
&\qquad \leqq \left| \int_{-A}^{A} f(y)dF_n(y) - \int_{-A}^{A}\left(\sum_{j=1}^{r} c_j f_{i_j}(y)\right)dF_n(y)\right| \\
&\qquad\quad + \left| \int_{-A}^{A}\left(\sum_{j=1}^{r} c_j f_{i_j}(y)\right)dF_n(y) - \int_{-A}^{A}\left(\sum_{j=1}^{r} c_j f_{i_j}(y)\right)dF(y)\right| \\
&\qquad\quad + \left| \int_{-A}^{A}\left(\sum_{j=1}^{r} c_j f_{i_j}(y)\right)dF(y) - \int_{-A}^{A} f(y)dF(y)\right|.
\end{aligned}$$

The first and last terms are bounded by $\left\|f - \sum_{j=1}^{r} c_j f_{ij}\right\|$, and the middle term by $\epsilon/3$. Hence

$$\left| \int_{-A}^{A} f(y)dF_n(y) - \int_{-A}^{A} f(y)dF(y)\right| < \epsilon.$$

The proof of the theorem is now immediate. For each real number t, let $\psi_n(t)=\int_{-A}^{A}e^{ity}dF_n(y)$, and let $\psi(t)=\int_{-A}^{A}e^{ity}dF(y)$. Then we have $\lim_{n\to\infty}\psi_n(t)=\psi(t)$ for every t, and the theorem follows from the continuity theorem for Fourier-Stieltjes transforms.

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REAL-VALUED MAPPINGS OF SPHERES

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This note concerns subsets Δ of the unit 2-sphere S such that (*) for each continuous real-valued mapping f of S there exists a rotation r of S with all points of $r(\Delta)$ having the same value under f. In 1942, Kakutani [3] proved that the set Δ of end points of an orthonormal set of 3 vectors has property (*). It was observed by de Mira Fernandes [5] that the same proof holds in case Δ is the set of vertices of any equilateral triangle. Yamabe and Yujobo [8] proved a generalization of Kakutani's theorem to n-space. Their method may be used to prove that the set Δ of vertices of an isosceles triangle has property (*) (this has been carried out in a Master's thesis of R. D. Johnson [2]). Here we prove that the set Δ of vertices of any triangle has property (*); the methods differ from both those of Kakutani and those of Yamabe and Yujobo.

Dyson [1] has proved that the set of vertices of a square centered at the origin has property (*); Livesay [4] has extended this to any rectangle centered at the origin. The problem of finding all such sets Δ having property (*) is unsolved.

THEOREM. *Let f be a continuous real-valued mapping of the sphere S and let $x_0, x_1, x_2 \in S$. There exists a rotation r with $f(r(x_0))=f(r(x_1))=f(r(x_2))$.*

We need the following lemma.

LEMMA. *Suppose that X is a unicoherent locally connected continuum, and that T is a map of period 2 on X without fixed points. Suppose A is a subset of X which* (i) *is closed in X,* (ii) *is invariant under T, and*

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