The proof of the theorem is now immediate. For each real number t, let $\psi_n(t) = \int_{-A}^{A} e^{ity} dF_n(y)$, and let $\psi(t) = \int_{-A}^{A} e^{ity} dF(y)$. Then we have $\lim_{n\to\infty} \psi_n(t) = \psi(t)$ for every t, and the theorem follows from the continuity theorem for Fourier-Stieltjes transforms.

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REAL-VALUED MAPPINGS OF SPHERES

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This note concerns subsets Δ of the unit 2-sphere S such that (*) for each continuous real-valued mapping f of S there exists a rotation r of S with all points of $r(\Delta)$ having the same value under f. In 1942, Kakutani [3] proved that the set Δ of end points of an orthonormal set of 3 vectors has property (*). It was observed by de Mira Fernandes [5] that the same proof holds in case Δ is the set of vertices of any equilateral triangle. Yamabe and Yujobo [8] proved a generalization of Kakutani's theorem to n-space. Their method may be used to prove that the set Δ of vertices of an isosceles triangle has property (*) (this has been carried out in a Master's thesis of R. D. Johnson [2]). Here we prove that the set Δ of vertices of any triangle has property (*); the methods differ from both those of Kakutani and those of Yamabe and Yujobo.

Dyson [1] has proved that the set of vertices of a square centered at the origin has property (*); Livesay [4] has extended this to any rectangle centered at the origin. The problem of finding all such sets Δ having property (*) is unsolved.

THEOREM. *Let f be a continuous real-valued mapping of the sphere S and let $x_0, x_1, x_2 \in S$. There exists a rotation r with $f(r(x_0)) = f(r(x_1)) = f(r(x_2))$.*

We need the following lemma.

LEMMA. *Suppose that X is a unicoherent locally connected continuum, and that T is a map of period 2 on X without fixed points. Suppose A is a subset of X which* (i) *is closed in X,* (ii) *is invariant under T, and*

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(iii) *separates* x *from* $T(x)$ *for all* x *from the complementary set. There exists a connected subset* B *of* A *having properties* (i), (ii), *and* (iii).

PROOF. There exists a set $B\subset A$ minimal with respect to possessing properties (i), (ii), and (iii). Since X is locally connected, there is a separation $X-B=P\cup Q$ with $T(P)=Q$. Consider any such separation. P is maximal with respect to the class of open sets R with $R\cap T(R)=\emptyset$. It follows from the maximality of P that (1) the interior of B is empty, and (2) $x\in\overline{P}\cap B$ if and only if $x\in\overline{Q}\cap B$. Hence $\overline{P}\cap\overline{Q}=B$.

A consequence of the minimality of B is that if $X=R\cup S$ where R is closed, $T(R)=S$, and $R\cap S\subset B$ then $R\cap S=B$. We show that if $X-B=P\cup Q$ is a separation with $P=T(Q)$ then $\overline{P}$ is connected. *Suppose* $\overline{P}=U\cup V$ *is a separation.* Then $\overline{Q}=T(U)\cup T(V)$; let $R=U\cup T(V)$ and $S=V\cup T(U)$. Then R is closed, $T(R)=S$, and $X=R\cup S$. Moreover $R\cap S=[U\cap T(U)]\cup[V\cap T(V)]=B$. But $U\cap T(U)\subset U\cap B$ and $V\cap T(V)\subset V\cap B$. Since $B=(U\cap B)\cup(V\cap B)$ is a separation, $U\cap T(U)=U\cap B$ and $V\cap T(V)=V\cap B$. Now $(U\cap T(U))\cup(U\cap T(V))=U\cap\overline{Q}=U\cap B$ and $U\cap T(U)=U\cap B$; it follows that $U\cap T(V)=\emptyset$ and similarly $V\cap T(U)=\emptyset$. Then $X=[U\cup T(U)]\cup[V\cup T(V)]$ is a separation of X. But X is connected; hence $\overline{P}$ is connected.

Now $X=\overline{P}\cup\overline{Q}$ where $\overline{P}$ and $\overline{Q}$ are continua. By the definition of unicoherence, $\overline{P}\cap\overline{Q}$ is connected. The lemma follows.

We now prove the theorem. Let X denote the group of rotations of S. Then X is homeomorphic with projective 3-space [6, p. 115]. Hence X is a unicoherent locally connected continuum. Let s_i be a rotation of order 2 interchanging x_0 and x_i, $i=1, 2$. Define $T_i: X\to X$ by $T_i(r)=r\circ s_i$, where $r\circ s_i$ is the composition s_i followed by r. Define real-valued mappings f_i of X, $i=1, 2$, by $f_i(r)=f(r(x_i))-f(r(x_0))$. A short computation shows that $f_i\circ T_i=-f_i$. Define A_i to be the set of all $r\in X$ with $f_i(r)=0$. Then A_i is closed and invariant under T_i. Since $f_i\circ T_i=-f_i$, if $r\in X-A_i$ then $f_i(r)$ and $f_i(T_i(r))$ have opposite signs, and hence A_i separates r from $T_i(r)$. By the lemma, there is a continuum B_i in X, invariant under T_i, and with B_i separating r from $T_i(r)$ for $r\in X-B_i$. Since X is locally connected, then $X-B_i=P_i\cup Q_i$ where P_i is open, $P_i\cap Q_i=\emptyset$, and $Q_i=T_i(P_i)$.

The transformations T_i are "translations" in the compact group X. They then preserve Haar measure on X, as does $T_1\circ T_2$. So if U is a nonempty open set in X with $U\neq X$ then $T_1(T_2(\overline{U}))\not\subset U$. For if $(T_1\circ T_2)\overline{U}\subset U$, then the measure of $(T_1\circ T_2)U$ is less than that of U, which contradicts the measure preserving property.

Suppose that $B_1 \cap B_2 = \emptyset$. Since B_2 is connected and disjoint from B_1, either $B_2 \subset P_1$ or $B_2 \subset Q_1$. Suppose the naming is so carried out that $B_2 \subset Q_1$; similarly suppose $B_1 \subset P_2$. Now, since B_1 is a continuum, $P_1 \cup B_1$ is connected. Since $P_1 \cup B_1$ intersects P_2 and does not intersect B_2, then $P_1 \cup B_1 \subset P_2$. Similarly $Q_2 \cup B_2 \subset Q_1$. Now

$$T_1(T_2(\overline{P}_1)) \subset T_1(T_2(P_1 \cup B_1)) \subset T_1(T_2(P_2)) \subset T_1(Q_2) \subset T_1(Q_1) = P_1.$$

We have seen that this is impossible. Hence $B_1 \cap B_2 \neq \emptyset$; suppose $r \in B_1 \cap B_2$. Then $f_1(r) = f_2(r) = 0$ and $f(r(x_0)) = f(r(x_1)) = f(r(x_2))$.

Bibliography

1. F. J. Dyson, *Continuous functions defined on spheres*, Ann. of Math. vol. 54 (1951) pp. 534–536.
2. R. D. Johnson, *Continuous real-valued functions on spheres*, Master's thesis, University of Virginia.
3. S. Kakutani, *A proof that there exists a circumscribing cube around any bounded closed convex set in* R^3, Ann. of Math. vol. 43 (1942) pp. 739–741.
4. G. R. Livesay, *On a theorem of F. J. Dyson*, Ann. of Math. vol. 59 (1954) pp. 227–229.
5. A. de Mira Fernandes, *Funzioni continue sopra una superficie sferica*, Portugaliae Mathematica vol. 5 (1946) pp. 132–134.
6. N. Steenrod, *The topology of fibre bundles*, Princeton, 1951.
7. G. T. Whyburn, *Analytic topology*, New York, 1941.
8. H. Yamabe and Z. Yujobo, *On the continuous function defined on a sphere*, Osaka Math. J. vol. 2 (1950) pp. 19–22.

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