ON MODIFIED BOREL METHODS

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1. **Introduction.** Given a series $\sum a_n$ with partial sums s_n it is possible to associate with it the Borel transforms

$$(1.1)\quad B(x; s_k) = e^{-x} \sum \frac{s_k x^k}{k!}, \quad B'(x; s_k) = \int_0^x e^{-t} a(t)dt, \quad a(t) = \sum \frac{a_k t^k}{k!}$$

for $x>0$. One says that B-lim $s_n = s$ [B'-lim $s_n = s$] if $\lim_{x\to\infty} B(x; s_k) = s$ [$\lim_{x\to\infty} B'(x; s_k) = s$]. The relations between these Borel methods B, B', and their behavior under change of index are known [**8**, p. 183; **6**; **7**].

Following a suggestion of R. P. Boas, Jr., we intend to study in this paper the modified Borel methods which arise when the continuous variable x in (1.1) is replaced by the discrete sequence of integers $n = 1, 2, \cdots$. The resulting methods shall be denoted by B_I and B_I', and our interest is to discuss the relations among the methods B, B_I, B', B_I' (which is done in §3) and the behavior of these methods under change of index (cf. §4). The methods B_I, B_I' show certain abnormalities in comparison with B, B'. For example, B-lim $s_n = s$ always implies B'-lim $s_n = s$, whereas B_I-lim $s_n = s$ implies B_I'-lim $s_n = s$ if $a_n = O(K^n)$ for $K < (\pi^2+1)^{1/2}$ and not for $K = (\pi^2+1)^{1/2}$.

Our results are based on two theorems on entire functions (§2). The first allows one to infer $f(x) \to s$ [$x \to +\infty$] from $f(n) \to s$ ($n = 1, 2, \cdots$) if the type of $f(z)$ is less than π, and is well known; the second allows one to infer $f(x) \cong se^x$ [$x \to +\infty$] from $f(n) \cong se^n$ ($n = 1, 2, \cdots$) if the type of $f(z)$ is less than $(\pi^2+1)^{1/2}$.

Finally, in §5 Cesàro-Borel methods are considered but the results there are incomplete, whereas the results in §3 and §4 are in a certain sense best possible.

2. **A theorem on functions of exponential type.** If $f(z)$ is regular in the angle $|\arg z| \leqq \alpha$ ($\alpha > 0$), it is said to be there of exponential type τ if for every $\epsilon > 0$, but for no $\epsilon < 0$, there exists a constant $M(\epsilon)$ such that

$$|f(z)| \leqq M(\epsilon) e^{(\tau+\epsilon)|z|} \qquad (|\arg z| \leqq \alpha).$$

The growth of $f(z)$ along the ray $\arg z = \phi$ ($|\phi| \leqq \alpha$) is described by the indicator function

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$$h_f(\phi) = \limsup_{r\to\infty} r^{-1} \log |f(re^{i\phi})| .$$

In §3 and §4 we meet the problem of going from the behavior of $f(n)$ $(n=1, 2, \cdots)$ to the behavior of $f(x)$ $(x\to+\infty)$. A well known theorem in this direction is[1]

THEOREM 1. *If $f(z)$ is regular and of exponential type in* $|\arg z| \leqq \alpha \leqq \pi/2$ $(\alpha>0)$, *and if*

$$h_f(\pm\alpha) < \pi \sin \alpha,$$

then $f(n)\to 1$ $(n=1, 2, \cdots)$ *implies* $f(x)\to 1$ $(x\to+\infty)$.

For our purposes we need an extension of Theorem 1 covering the case $f(n)\cong e^n$ instead of $f(n)\to 1$.

THEOREM 2. *If $f(z)$ is regular and of exponential type in* $|\arg z| \leqq \alpha \leqq \pi/2$ $(\alpha>0)$, *and if*

$$h_f(\pm\alpha) < \pi \sin \alpha + a \cos \alpha \qquad (a \geqq 0), \tag{2.1}$$

then

(2.2) $f(n)\cong n^k\cdot e^{an}$ $(n=1, 2, \cdots)$ *implies* $f(x)\cong x^k\cdot e^{ax}$ $(x\to+\infty)$ $(k\geqq 0)$.

In particular, (2.2) *is true if $f(z)$ is regular and of exponential type* $\tau<(\pi^2+a^2)^{1/2}$ *in* $\Re(z)\geqq 0$; *for* $\tau=(\pi^2+a^2)^{1/2}$ *this is false.*

PROOF. Consider $g(z)=f(z)e^{-az}(z+1)^{-k}$ in $|\arg z|\leqq\alpha$, where $(z+1)^k$ is assumed to be >0 for $z>0$. For the indicator function of $g(z)$ on $\arg z=\pm\alpha$ we have

$$\begin{aligned} h_g(\pm\alpha) &= \limsup_{r\to\infty} r^{-1}\log |g(re^{\pm i\alpha})| \\ &= \limsup_{r\to\infty} r^{-1}\log |f(re^{\pm i\alpha})| - a\cos\alpha < \pi\sin\alpha \end{aligned}$$

by (2.1), and hence, by Theorem 1, $g(n)\to 1$ $(n=1, 2, \cdots)$ implies $g(x)\to 1$ $(x\to+\infty)$.

If, in particular, $f(z)$ is of exponential type $\tau<(\pi^2+a^2)^{1/2}$ in $\Re(z)\geqq 0$, we choose α such that $tg\alpha=\pi/a$, so that

$$\pi\sin\alpha + a\cos\alpha = \pi/\sin\alpha = (\pi^2+a^2)^{1/2} > \tau \geqq h_f(\pm\alpha),$$

i.e. hypothesis (2.1) is fulfilled and hence (2.2) follows.

For the last part of the theorem consider $f(z)=e^{az}(\sin\pi z+1)$.

[1] Theorem 1 is implicitly contained in Cartwright [2], explicitly in Macintyre [9, p. 16]. See also Pfluger [12, pp. 312–314], Duffin-Schaeffer [5, pp. 142–143] and Boas [1, p. 180].

3. Relations between the Borel methods. Now we are going to consider the methods of summability which associate with a given series the following transformations:

$$B:\ e^{-x}\sum \frac{s_k x^k}{k!}\ (x>0); \qquad B':\ \int_0^x e^{-t}a(t)dt\ (x>0);$$

$$a(t)=\sum \frac{a_k t^k}{k!},$$

$$B_I:\ e^{-n}\sum \frac{s_k n^k}{k!}\ (n=1,2,\cdots);\ B_I':\ \int_0^n e^{-t}a(t)dt\ (n=1,2,\cdots).$$

The B- and B'-transformations are connected by the formal relation (Hardy [8, p. 182])

$$B(x;s_k)=B(x;a_k)+B'(x;s_k), \text{ i.e.}$$

$$e^{-x}\sum \frac{s_k x^k}{k!}=e^{-x}\sum \frac{a_k x^k}{k!}+\int_0^x e^{-t}a(t)dt. \tag{3.1}$$

The problem of this paragraph is to investigate the relative strength of the above Borel methods. For two summability methods V_1 and V_2 we use the notation $V_1\rightarrow V_2$, if V_1-lim $s_n=s$ implies always V_2-lim $s_n=s$.

The following relations are trivial or known.

(3.2) $B\rightarrow B_I$ and $B'\rightarrow B_I'$.

(3.3) $B\rightarrow B'$ (Hardy [8, p. 183]).

(3.4) $B'\rightarrow B$ if $a_n=O(K^n)$ for some $K>0$ (Gaier [6, p. 455]). This becomes false if $a_n=O(K^n)$ is replaced by $a_n=O(n^{\epsilon n}K^n)$ (ϵ arbitrary >0) (Gaier [7]).

Our new results about the relations between the Borel methods are summarized in

THEOREM 3. (1) *$B_I\rightarrow B$, $B_I\rightarrow B'$, and $B_I\rightarrow B_I'$, if $a_n=O(K^n)$ for $K<(\pi^2+1)^{1/2}$, but not for $K=(\pi^2+1)^{1/2}$.*

(2) $B_I'\rightarrow B'$, $B_I'\rightarrow B$, and $B_I'\rightarrow B_I$, if $a_n=O(K^n)$ for $K<(\pi^2+1)^{1/2}$, but not for $K=(\pi^2+1)^{1/2}$.

Note, in particular, that there is no analogy to (3.3) for the methods B_I and B_I'.

PROOF. (1) (a) $B_I\rightarrow B$. (i) If $a_n=O(K^n)$ $(K<(\pi^2+1)^{1/2})$, then $|s_n|\leqq MK'^n$ $(K'<(\pi^2+1)^{1/2})$ and the entire function $\phi(z)=\sum s_n z^n/n!$ satisfies the estimation

$$|\phi(z)|\leqq M\sum \frac{K'^n|z|^n}{n!}=Me^{K'|z|},$$

i.e. it is of type $\tau<(\pi^2+1)^{1/2}$. Therefore the assumption $\phi(n)\cong A\cdot e^n$ $(n=1, 2, \cdots)$ implies, by Theorem 2, $\phi(x)\cong A\cdot e^x$ $(x\to+\infty)$, i.e. B-lim $s_n=A$.

(ii) Define s_n by $\sum(s_n z^n/n!)=e^z(\sin \pi z+1)$. Then $(\alpha) B_I$-lim $s_n=1$, but not B-lim $s_n=1$. (β) One finds immediately

$$s_n = 1 + (1/2i)\{(1+i\pi)^n - (1-i\pi)^n\},$$

so that $s_n=O((\pi^2+1)^{n/2})$ and also $a_n=O((\pi^2+1)^{n/2})$ are fulfilled.

(b) $B_I\to B'$. (i) The assumption about the a_n implies (Case (a) and (3.3))

$$B_I \to B \to B'.$$

(ii) Define s_n as above. Then (α) B_I-lim $s_n=1$, but not B'-lim $s_n=1$; otherwise B'-lim $s_n=1$ would by (3.4) imply B-lim $s_n=1$ which is false. (β) $a_n=O((\pi^2+1)^{n/2})$ is fulfilled.

(c) $B_I\to B_I'$. (i) The assumption about the a_n implies (Case (b) and (3.2))

$$B_I \to B' \to B_I'.$$

(ii) Define a_n by

$$\int_0^z e^{-t}a(t)dt = \sin(\pi z+\alpha); \qquad tg\alpha = -\pi.$$

Then (α) B_I-lim $s_n=0$. For, by the relation (3.1), we have

$$B(x; s_k) = \frac{d}{dx}\sin(\pi x+\alpha) + \sin(\pi x+\alpha),$$

which, taken at $x=n$ $(n=1, 2, \cdots)$, becomes

$$B(n; s_k) = \cos \pi n(\sin\alpha + \pi\cos\alpha) = 0 \qquad (n = 1, 2, \cdots).$$

On the other hand B_I'-lim s_n does not exist. (β) We have

$$a(t) = e^t\cdot\pi\cos(\pi t+\alpha) = \sum \frac{a_k t^k}{k!},$$

from which $a_n=O((\pi^2+1)^{n/2})$ is immediate.

(2) (a) $B_I'\to B'$. (i) If $a_n=O(K^n)$ $(K<(\pi^2+1)^{1/2})$, then $a(t)$ is an entire function of exponential type $\tau<(\pi^2+1)^{1/2}$. If therefore $g(z)=e^{-z}a(z)$, we have for the indicator function of $g(z)$ taken for the rays $\arg z=\pm\alpha$ $(tg\alpha=\pi)$

$$h_g(\pm\alpha) = h_a(\pm\alpha) - \cos\alpha < (\pi^2+1)^{1/2} - \cos\alpha = \pi\sin\alpha,$$

and hence for the function $\phi(z)=\int_0^z e^{-t}a(t)dt$

$$h_\phi(\pm\alpha) < \pi \sin \alpha,$$

so that an application of Theorem 1 infers $\phi(x)\to A(x\to+\infty)$ from $\phi(n)\to A$ $(n=1, 2, \cdots)$.

(ii) Define a_n by $\int_0^z e^{-t}a(t)dt=\sin \pi z$. Obviously B_I'-lim $s_n=0$, but not B'-lim $s_n=0$. The validity of $a_n=O((\pi^2+1)^{n/2})$ is again immediate.

(b) $B_I'\to B$. (i) The assumption about the a_n implies (Case (a) and (3.4))

$$B_I' \to B' \to B.$$

(ii) Define a_n as in (2) (a). B-lim $s_n=0$ cannot hold since B'-lim s_n does not exist.

(c) $B_I\to B_I$. (i) By the preceding case $B_I'\to B\to B_I$.

(ii) Define a_n as in (2) (a). By (3.1), the B-transform of the corresponding sequence s_n is $\sin \pi x+\pi \cos \pi x$, so that $B_I(n; s_k)=\pm\pi$ $(n=1, 2, \cdots)$.

4. On the change of index for the methods B_I and B_I'. We consider the two series

$$\sum a_k = a_0 + a_1 + a_2 + \cdots \qquad \text{with partial sums } s_n$$

and

$$\sum b_k = 0 + a_0 + a_1 + \cdots \qquad \text{with partial sums } t_n.$$

The problem is to determine under what conditions

$$V\text{-lim } s_n = s \quad \text{implies} \quad V\text{-lim } t_n = s \tag{4.1.a}$$

or

$$V\text{-lim } t_n = s \quad \text{implies} \quad V\text{-lim } s_n = s, \tag{4.1.b}$$

where V is one of the methods B_I, B_I'.[2]

In addition to (3.1) we shall need the relations

$$B'(x; s_k) = B(x; t_k) \qquad \text{(Hardy [8, p. 182])} \tag{4.2}$$

and

$$B'(x; s_k) = B(x; b_k) + B'(x; t_k). \tag{4.3}$$

[2] This problem has been treated for other methods of summability; cf. Doetsch [**4**], p. 464 ff. for B; Doetsch [**3**], for C_kB; Gaier [**6**] and [**7**] for B; Meyer-König [**10**, p. 270] and Meyer-König and Zeller [**11**, pp. 348–349] for T_α, S_α, T_α'; Wollan [**13**, p. 583] for Euler summability of double series.

Note that $B(x;\ b_k)=(d/dx)B'(x;\ t_k)$. The proof of (4.3) follows from[8]

$$B'(x;t_k) = \int_0^x e^{-t}b(t)dt = -e^{-t}b(t)\Big|_0^x + \int_0^x e^{-t}a(t)dt = -B(x;b_k) + B'(x;s_k).$$

THEOREM 4. *If V is one of the methods B_I, B_I', both statements* (4.1.a) *and* (4.1.b) *are correct if $a_n=O(K^n)$ for $K<(\pi^2+1)^{1/2}$, but not for $K=(\pi^2+1)^{1/2}$.*

Note, in particular, that there is no analogy to the fact that (4.1.a) holds for $V=B$ without restriction of the a_n.

PROOF. (1) $V=B_I$. (a) By (4.2), B_I-lim $t_n=s$ if and only if B_I'-lim $s_n=s$, which follows from B_I-lim $s_n=s$ if $a_n=O(K^n)$ for $K<(\pi^2+1)^{1/2}$, but not for $K=(\pi^2+1)^{1/2}$ (Theorem 3, 1c).

(b) Again, B_I-lim $t_n=s$ if and only if B_I'-lim $s_n=s$, which implies B_I-lim $s_n=s$ if $a_n=O(K^n)$ for $K<(\pi^2+1)^{1/2}$, but not for $K=(\pi^2+1)^{1/2}$ (Theorem 3, 2c).

(2) $V=B_I'$. (a) (i) If $a_n=O(K^n)(K<(\pi^2+1)^{1/2})$, B_I'-lim $s_n=s$ implies B'-lim $s_n=s$ so that by (4.3) $\phi(x)+\phi'(x)\to s(x\to+\infty)$ $[\phi'(x)=B(x;\ b_k)]$, and consequently (Hardy [8, p. 107]) $\phi(x)\to s(x\to+\infty)$, i.e. B_I'-lim $t_n=s$.

(ii) Define b_n by $\int_0^x e^{-t}b(t)dt=\sin\ (\pi x+\alpha)$ with $tg\alpha=-\pi$ and proceed as in Theorem 3, 1(c) (ii). We get B_I'-lim $s_n=0$ whereas B_I'-lim t_n does not exist, although $a_n=O((\pi^2+1)^{n/2})$.

(b) (i) If $a_n=O(K^n)(K<(\pi^2+1)^{1/2})$, B_I'-lim $t_n=s$ implies B'-lim $t_n=s$ (Theorem 2), and since $B'(z;\ t_k)$ is an entire function of exponential type tending to s as $z\to+\infty$, its derivative $e^{-z}b(z)=B(z;\ b_k)$ tends to zero as $z\to+\infty$ (Boas [1, p. 212] and Gaier [6, p. 454]) which, by (4.3), implies B_I'-lim $s_n=s$.

(ii) Define b_n by $\int_0^x e^{-t}b(t)dt=\sin\ \pi x$. Then B_I'-lim $t_n=0$, but not B_I'-lim $s_n=0$, although $a_n=O((\pi^2+1)^{n/2})$.

5. **Cesàro-Borel methods.** Doetsch [3] was the first to consider the Cesàro-Borel transform

$$C_kB(x;\ s_k) = kx^{-k}\int_0^x B(t;\ s_k)(x-t)^{k-1}dt \qquad (k>0,\ x\geqq 0),$$

and in view of our results in §3 one can ask what relations there are

[8] Let $b(t)=\sum(b_nt^n/n!)$, so that $b'(t)=a(t)$.

for example between the methods C_kB and C_kB_I (C_k = matrix method in the latter case). It is not surprising that in general

$$(5.1)\qquad C_kB_I\text{-lim } s_n = s \quad \textit{does not imply} \quad C_kB\text{-lim } s_n = s;$$

however, also

$$(5.2)\qquad C_kB\text{-lim } s_n = s \textit{ does not imply } C_kB_I\text{-lim } s_n = s.$$

Equivalent to the problem raised is, of course, under what conditions for an entire function $f(z)$ does

$$C_k\text{-lim } f(n) = s \quad \text{imply} \quad C_k\text{-lim } f(x) = s$$

and conversely. For $k=1$ the statement (5.1) follows from consideration of $f(z) = z \sin \pi z$, whereas for the proof of (5.2) we take an entire function $f(z)$ of exponential type $(<2\pi+\epsilon)$ which is, for $x>0$,

$$f(x) = x^{1/2} \cos 2\pi x + o(1).[4]$$

Then obviously C_1-lim $f(n) = +\infty$, but C_1-lim $f(x) = 0$. The author has no contribution towards the solution of this problem.

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[4] To obtain such a function apply Macintyre's lemma [1, p. 80] to $f(z) = z^{1/2} \cdot \cos 2\pi z$; $f(z/2+\epsilon)$ is of type $<\pi$ in $\Re(z) \geqq 0$.