

DEGENERATE SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS¹

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Introduction. We consider a partial differential equation of second order:

$$(1) \quad L(u) = \sum_{i,j=1}^n a_{ij}(p_1, p_2, \dots, p_n) \partial^2 u / \partial x_i \partial x_j = 0$$

with n independent variables $x = (x_1, x_2, \dots, x_n)$, the coefficients are functions of the first partial derivatives, $a_{ij} = a_{ji}$, $p_i = \partial u / \partial x_i$, but they do not depend on x . The equation of motion of compressible fluids is, for instance, of the form (1). Certain types of degenerate compressible flows, namely the simple and double waves, are of practical importance and have been treated by various authors, [1; 4; 6]. An " s -tuple wave" u , if it exists, is a solution of (1) whose first partial derivatives satisfy $n-s$ functional relations among themselves (for this notion, see [2, pp. 76-78]). Let the relations be represented by

$$(2) \quad p_\alpha = F^\alpha(p_1, p_2, \dots, p_s), \quad \alpha = s+1, s+2, \dots, n,$$

where the functions F^α are assumed to have continuous second derivatives. A necessary and sufficient condition for the existence of s -tuple waves is that F^α satisfy a system of partial differential equations (15) of second order and degree $s-1$ in the variables p_1, p_2, \dots, p_s . These equations are derived in §1, in a very simple manner, by the use of elementary contact transformations. In the case of $s=2$ (see (19)), and the case of $s=n-1$ (centered wave, see (22)), there are as many differential equations as the number of unknown functions. The existence of solutions of (19) leads to hyperbolic double waves of (1). Conversely, when a_{ij} are constants, we may make use of the explicit formulas for solutions of (1) to solve the nonlinear differential equations (19). This is done in §3.

1. The indices i and j are to run from 1 to n , k and l from 1 to s , and α and β from $s+1$ to n , $1 \leq s \leq n$; terms with repeated indices are to be summed. We introduce new independent variables $x' = (x'_1, x'_2, \dots, x'_n)$ and dependent variable $u'(x')$ by

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$$(3) \quad x'_k = p_k, \quad x'_\alpha = x_\alpha, \quad u' = u - p_k x_k.$$

The following simple calculation:

$$\begin{aligned} du' &= du - p_k dx_k - x_k dp_k = p_i dx_i - p_k dx_k - x_k dp_k \\ &= p_\alpha dx'_\alpha + (-x_k) dx'_k, \end{aligned}$$

gives the first partial derivatives $p'_i = \partial u' / \partial x'_i$ by

$$(4) \quad p'_k = -x_k, \quad p'_\alpha = p_\alpha.$$

(3) and (4) together form an elementary contact transformation. The equation $dp'_k = p'_{kj} dx'_j$ can be written as follows,

$$(5) \quad -dx_k = p'_{k\beta} dx_\beta + p'_{kl} dp_l.$$

We assume that the determinant $D = |p'_{kl}|$ of s rows and columns is different from zero. Then we can solve for dp_l in (5) and obtain

$$(6) \quad D dp_l = -Q_{lk}(p'_{k\beta} dx_\beta + dx_k),$$

where Q_{lk} is the cofactor of p'_{lk} in the determinant D . The remaining $n-s$ differentials dp_α are found by making use of our assumption on the existing functional relations (2):

$$(7) \quad D dp_\alpha = D dp_l \partial F^\alpha / \partial p_l = -Q_{lk}(p'_{k\beta} dx_\beta + dx_k) \partial F^\alpha / \partial p_l.$$

Since $dp_i = p_{ij} dx_j$, one can derive from (6) and (7) the second derivatives p_{ij} in terms of D^{-1} , Q_{lk} , p'_{kl} , $p'_{k\alpha}$ and $\partial F^\alpha / \partial p_k$.

On the other hand we can easily establish the following:

$$(8) \quad u' = x'_\alpha F^\alpha(x'_1, x'_2, \dots, x'_s) + G(x'_1, x'_2, \dots, x'_s),$$

$$(9) \quad p'_{k\alpha} = \partial F^\alpha / \partial x'_k,$$

$$(10) \quad p'_{kl} = x'_\alpha \partial^2 F^\alpha / \partial x'_l \partial x'_k + \partial^2 G / \partial x'_l \partial x'_k,$$

where G is some function independent of x'_α . Indeed from $p'_{k\alpha} = p'_{\alpha k}$ and $p'_\alpha = p_\alpha$ it follows (9) by the differentiation of $p_\alpha = F^\alpha$ with respect to x_k . The right-hand side of (9) is independent of x'_α , from which we conclude that

$$(11) \quad p'_k = x_\alpha \partial F^\alpha / \partial x'_k + G_k(x'_1, x'_2, \dots, x'_s)$$

holds. Due to $p'_{kl} = p'_{lk}$ we must have $\partial G_k / \partial x'_l = \partial G_l / \partial x'_k$, thus it is justified to write $G_k = \partial G / \partial x'_k$. From (11) follow (8) and (10).

We substitute (9) and (10) respectively into the right-hand side of (6) and (7), and use (3) to replace x'_k by p_k , and x'_α by x_α . The following formulas are immediately verified:

$$(12) \quad Dp_{ik} = -Q_{ik}, \quad Dp_{i\beta} = -Q_{ik}\partial F^\beta/\partial p_k,$$

$$Dp_{\alpha\beta} = -Q_{ik}\partial F^\alpha/\partial p_i\partial F^\beta/\partial p_k.$$

The above expressions for p_{ij} are now inserted into $L(u)$. Because $L(u)$ is homogeneous of second order, the factor D can be cancelled, and (1) becomes

$$(13) \quad (a_{ik} + 2a_{i\beta}\partial F^\beta/\partial p_k + a_{\alpha\beta}\partial F^\alpha/\partial p_i\partial F^\beta/\partial p_k)Q_{ik} = 0.$$

Note that the coefficients of Q_{ik} in (13) are functions of p_k and are independent of x'_α .

Now, each Q_{ik} is a cofactor in D and is therefore a sum of $(s-1)!$ products $\pm \prod p'_{ik}$. By substituting (10) into each factor and by carrying out the multiplication, each product becomes an inhomogeneous polynomial of x'_α of degree $s-1$. Each monomial is of the form $x_{s+1}^{m_{s+1}} x_{s+2}^{m_{s+2}} \cdots x_n^{m_n}$ with non-negative exponents, $0 \leq m_{s+1} + m_{s+2} + \cdots + m_n \leq s-1$. There are altogether $C(n-1, s-1)$ distinct monomials, each corresponds to an arrangement of the exponents $(m) = (m_{s+1}, m_{s+2}, \cdots, m_n)$. We add up all the $(s-1)!$ products in Q_{ik} and collect the terms with the same monomial, we find

$$(14) \quad Q_{ik} = \sum_{(m)} P_{ik}^{(m)} x_{s+1}^{m_{s+1}} x_{s+2}^{m_{s+2}} \cdots x_n^{m_n},$$

where the summation stretches over $C(n-1, s-1)$ terms. Each $P_{ik}^{(m)}$ is a homogeneous polynomial of $s-1$ degree in the second derivatives of F^α and G with integer coefficients.

THEOREM 1. *A necessary and sufficient condition for the existence of a degenerate solution u , of the type of an s -tuple wave, with nonvanishing determinant $|\partial^2 u/\partial x_k \partial x_l|$, is the following: the $n-s+1$ functions F^α and G satisfy a system of $C(n-1, s-1)$ differential equations of second order and degree $s-1$, in the variables p_k ,*

$$(15) \quad (a_{ik} + 2a_{i\beta}\partial F^\beta/\partial p_k + a_{\alpha\beta}\partial F^\alpha/\partial p_i\partial F^\beta/\partial p_k)P_{ik}^{(m)} = 0,$$

with

$$(16) \quad |x_\alpha \partial^2 F^\alpha/\partial p_i \partial p_k + \partial^2 G/\partial p_i \partial p_k| \neq 0.$$

PROOF. The condition is necessary. By the first equation in (12), $|p_{ik}| \neq 0$ leads to $D \neq 0$ and hence (16). We put (14) into (13) and separate the distinct monomials from one another, which results in splitting up (13) into $C(n-1, s-1)$ differential equations (15) as mentioned in the theorem.

Conversely, if one substitutes a solution of (15) into (8) and (3), one finds

$$(17) \quad u = x_\alpha F^\alpha(p_1, p_2, \dots, p_s) + G(p_1, p_2, \dots, p_s) + p_k x_k.$$

Write (11) in the form

$$(18) \quad -x_k = x_\alpha \partial F^\alpha / \partial p_k + \partial G / \partial p_k,$$

then due to (16) one may solve (18) for p_k in terms of x . Then (17) leads to a solution $u(x)$ which is a degenerate solution of (1).

2. When $s=2$, the case of double waves, there are $n-1$ equations and the same number of unknown functions $F^\alpha(p_1, p_2)$ and $G(p_1, p_2)$, $\alpha=3, 4, \dots, n$. The differential equations can be written in the following form:

$$(19) \quad AF_{11}^\alpha - 2BF_{12}^\alpha + CF_{22}^\alpha = 0, \quad \alpha = 3, 4, \dots, n,$$

$$(20) \quad AG_{11} - 2BG_{12} + CG_{22} = 0,$$

where the subindices stand for differentiations with respect to p_1 and p_2 . The coefficients are the same for all the equations:

$$(21) \quad \begin{aligned} A &= a_{22} + 2a_{2\beta}F_2^\beta + a_{\alpha\beta}F_2^\alpha F_2^\beta, \\ C &= a_{11} + 2a_{1\beta}F_1^\beta + a_{\alpha\beta}F_1^\alpha F_1^\beta, \\ B &= a_{12} + (a_{1\beta}F_2^\beta + a_{2\beta}F_1^\beta) + a_{\alpha\beta}F_1^\alpha F_2^\beta. \end{aligned}$$

Centered waves are those with $G=0$. Since the coefficients in (21) do not depend on G , for the study of double waves, one may restrict oneself to centered ones.

For centered waves with $s=n-1$, there is only one unknown function F . Equation (14) is simply $Q_{lk} = P_{lk}x_n^{n-2}$. We form the determinant $|F_{lk}|$ of $n-1$ rows and columns of the second derivatives of F . It is seen that $-P_{lk}$ is the cofactor f_{lk} of F_{lk} in that determinant. Hence (15) becomes for the $n-1$ -tuple centered waves the following:

$$(22) \quad \sum_{l,k=1}^{n-1} (a_{lk} + 2a_{ln}F_k + a_{nn}F_l F_k) f_{lk} = 0.$$

An interesting example is given by $L(u) = \nabla^2 u(x_1, x_2, x_3)$. The corresponding equation (22) is the differential equation of minimal surfaces. This fact is well-known, for other proofs see for instance [3, pp. 44-46] and [5].

3. Let $s_1(\tau)$, $s_2(\tau)$, $f^\alpha(\tau)$, $f_1^\alpha(\tau)$ and $f_2^\alpha(\tau)$ be continuously differentiable functions of τ , for $0 \leq \tau \leq 1$. It is assumed that

$$(23) \quad \dot{f}^\alpha = f_1^\alpha \dot{s}_1 + f_2^\alpha \dot{s}_2, \quad \alpha = 3, 4, \dots, n,$$

hold, where the dot stands for differentiation with respect to τ . If we replace F_k^α and F_k^β in (21) by f_k^α and f_k^β , and moreover, replace p_1 , p_2 and p_α in the argument of a_{ij} by s_1 , s_2 and f^α respectively, then the coefficients A , B and C become functions of τ , to be denoted by \overline{A} , \overline{B} and \overline{C} . We assume that

$$(24) \quad \overline{B}^2 - \overline{A}\overline{C} > 0,$$

and furthermore,

$$(25) \quad \overline{A}(\dot{s}_2)^2 - 2\overline{B}(\dot{s}_1)(\dot{s}_2) + \overline{C}(\dot{s}_1)^2 \neq 0.$$

Now, we consider the following two initial value problems. Problem I concerns system (19) for the unknown functions $F^\alpha(p_1, p_2)$. The initial curve in the (p_1, p_2) plane is given by

$$(26) \quad \Gamma: p_1 = s_1(\tau), \quad p_2 = s_2(\tau).$$

The initial values are

$$(27) \quad F^\alpha = f, \quad F_1^\alpha = f_1^\alpha, \quad F_2^\alpha = f_2^\alpha.$$

To formulate problem II, which concerns equation (1) for the unknown function $u(x)$, we have to define the initial manifold M_{n-1} in the x -space. One constructs first a one parametric family of hyperplanes, each of $n-2$ dimensions, by

$$(28) \quad E_{n-2}(\tau): -x_1 = x_\alpha f_1^\alpha(\tau), \quad -x_2 = x_\alpha f_2^\alpha(\tau).$$

The quadratic form

$$(29) \quad J(\tau) = \overline{A}\xi_1^2 + 2\overline{B}\xi_1\xi_2 + \overline{C}\xi_2^2$$

with $\xi_1 = x_\alpha \dot{f}_1^\alpha$ and $\xi_2 = x_\alpha \dot{f}_2^\alpha$, defines by $J=0$, a pair of hyperplanes each of dimensions $n-1$. The intersection of $E_{n-2}(\tau)$ with $J(\tau)=0$ is of the dimension $n-3$, unless all \dot{f}_1^α and \dot{f}_2^α are equal to zero, which case we exclude. Hence we may form $E'_{n-2} = E_{n-2} \cap (J>0)$ and $E''_{n-2} = E_{n-2} \cap (J<0)$. The initial manifold M_{n-1} is either the union of E'_{n-2} or of E''_{n-2} for $0 \leq \tau \leq 1$. The following values are assigned on M_{n-1} :

$$(30) \quad \begin{aligned} u &= s_1x_1 + s_2x_2 + x_\alpha f^\alpha, \\ \partial u / \partial x_1 &= s_1(\tau), \quad \partial u / \partial x_2 = s_2(\tau), \quad \partial u / \partial x_\alpha = f^\alpha(\tau). \end{aligned}$$

Due to (23) and (28) the data are compatible. The first partial derivatives of u are constant along each E_{n-2} in M_{n-1} .

THEOREM 2. *The solution of problem I, which exists uniquely in a neighborhood of Γ , gives a solution of problem II in the neighborhood of M_{n-1} . Conversely, let us assume that for the differential equation (1) problem II has an unique solution $u = U(x)$, which can be obtained explicitly. Then by the simple means of differentiation and elimination, one can find from $U(x)$ the solution of problem I.*

PROOF. (a) Since (19) is hyperbolic with respect to the data (27), and since Γ is not tangent to the characteristics, it follows the existence of an unique solution of problem I, [3, p. 333]. In order to obtain $u(x)$ from $F^\alpha(p_1, p_2)$ by (17) and (18), we have to exclude from the x -space those points for which the determinant (16): $J^* = (x_\alpha F_{12}^\alpha)^2 - (x_\alpha F_{11}^\alpha)(x_\alpha F_{22}^\alpha) \neq 0$. Now, the values of F_u^α along the initial curve Γ can be calculated. One verifies, after some computations, that $J^* = J(\tau)$ holds on Γ , with $J(\tau)$ given in (29). By the definition of M_{n-1} , it follows the first part of the theorem.

(b) Under the assumption of uniqueness, the solution $U(x)$ must be the same as that obtained through the solution of problem I, in the neighborhood of M_{n-1} . By Theorem 1, the determinant

$$(\partial^2 u / \partial x_1 \partial x_2)^2 - \partial^2 u / \partial x_1^2 \partial^2 u / \partial x_2^2 \neq 0$$

on M_{n-1} . We can therefore solve for x_1 and x_2 in terms of p_1, p_2 and x_α , by applying the theorem of implicit functions on $p_1 = \partial U / \partial x_1$, $p_2 = \partial U / \partial x_2$. Replacing x_1 and x_2 in $\partial U / \partial x_\alpha$ we obtain F^α .

The assumption made in the above theorem is satisfied, when the coefficients a_{ij} are constants, $L(u)$ is totally hyperbolic and the initial manifold is "space like." By (28) one finds the normal directions on M_{n-1} :

$$\frac{\partial x_1}{\partial v} : \frac{\partial x_2}{\partial v} : \dots : \frac{\partial x_\alpha}{\partial v} : \dots = (x_\alpha f_2^\alpha) : (-x_\alpha f_1^\alpha) : \dots : (x_\beta f_2^\beta f_1^\alpha - x_\beta f_1^\beta f_2^\alpha) : \dots$$

One inserts these expressions into $\bar{L} \equiv a_{ij} \partial x_i / \partial v \partial x_j / \partial v$. By a simple computation it is seen that $\bar{L} = J(\tau)$ of (29). Thus the initial manifold is "space like," if $J(\tau)$ has a definite sign.

To illustrate the theorem, we give briefly a simple example. Consider the following nonlinear differential equation in two variables: $(a^2 + F_2^2)F_{11} - 2F_1F_2F_{12} + (-c^2 + F_1^2)F_{22} = 0$ for $F(p_1, p_2)$ with the initial values 1°. $F(p_1, 0) = 0$, $F_2(p_1, 0) = f_2(p_1)$, and 2°. $F(0, p_2) = f(p_2)$, $F_1(0, p_2) = 0$. In both cases the solutions can be obtained by solving the wave equation in three independent variables: $a^2 u_{xx} + u_{yy} - c^2 u_{tt} = 0$. We assume that the given functions $f_2(p_1)$ and df/dp_2 can be solved implicitly by ϕ and ψ so that $\phi(f_2(p_1)) \equiv p_1$ and $\psi(df/dp_2) \equiv p_2$.

The initial data to be imposed on the wave equation are 1°. $u(x, y, 0) = 0$, $u_t(x, y, 0) = \phi(-x/y)$, 2°. $u(x, y, 0) = y \cdot f(\psi(-x/y)) + x \cdot \psi(-x/y)$, $u_t(x, y, 0) = 0$. The initial manifold here is the plane $t = 0$. If the procedure of elimination among the first partial derivatives is difficult to carry out, we have in the explicit formulas $p_1 = u_x(x, y, t)$, $p_2 = u_t(x, y, t)$, $p_3 = u_y(x, y, t)$ a parametric representation of the function $p_3 = F(p_1, p_2)$.

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