REAL LINEAR CHARACTERS OF THE SYMPLECTIC MODULAR GROUP

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1. The symplectic modular group Γ_{2n} consists of all integral $2n \times 2n$ matrices $\mathfrak{M}$ for which $\mathfrak{M}\mathfrak{F}\mathfrak{M}' = \mathfrak{F}$, where

$$\mathfrak{F} = \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}.$$

In order to determine all possible automorphisms of Γ_{2n} [1],[1] it is necessary to find all real linear characters of Γ_{2n}, that is, all homomorphisms into $\{\pm 1\}$. In this note we prove that Γ_{2n} has no nontrivial real linear characters for $n > 2$, while Γ_2 and Γ_4 each have exactly one nontrivial real linear character. We shall also determine Γ'_{2n}, the commutator subgroup of Γ_{2n}.

We define the symplectic direct sum $\mathfrak{M}_1 * \mathfrak{M}_2$ by

$$\mathfrak{M}_1 * \mathfrak{M}_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} * \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix}.$$

Set

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define[2]

$$U_0 = S \dotplus I^{(n-2)}, \qquad U_1 = V \dotplus I^{(n-2)}, \qquad U_2 = T \dotplus I^{(n-2)}.$$

Then [2] Γ_{2n} is generated by $\mathfrak{R}_i = U_i \dotplus U_i'^{-1}$ $(i = 0, 2)$, $\mathfrak{T}_0 = T * I^{2(n-1)}$, $\mathfrak{S}_0 = S * I^{2(n-1)}$, and their conjugates. When $n = 1$, the $\mathfrak{R}_i$ are superfluous. Next we remark that

$$(1) \qquad \mathfrak{S}_0\mathfrak{T}_0 = (ST) * I = (ST)^{-2} * I,$$

$$(2) \qquad \mathfrak{R}_0\mathfrak{R}_2 = U_3 \dotplus U_3'^{-1}, \text{ where } U_3 = ST \dotplus I = (ST \dotplus I)^{-2},$$

$$(3) \qquad \mathfrak{T}_0 = \mathfrak{R}_1\mathfrak{R}_2 \cdot \mathfrak{R}_0\mathfrak{T}_0\mathfrak{R}_0^{-1}\mathfrak{T}_0^{-1} \cdot (\mathfrak{R}_1\mathfrak{R}_2)^{-1} \cdot \mathfrak{S}_0\mathfrak{R}_1 \cdot \mathfrak{R}_2 \cdot (\mathfrak{S}_0\mathfrak{R}_1)^{-1}.$$

Presented to the Society, December 29, 1954 under the title *Characters of the symplectic modular group*; received by the editors November 13, 1954 and, in revised form, November 24, 1954.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] $A \dotplus B$ denotes the direct sum of the matrices A and B.

Therefore if θ is any real linear character of Γ_{2n}, we must have

$$\theta(\mathfrak{R}_0) = \theta(\mathfrak{R}_2) = \theta(\mathfrak{T}_0) = \theta(\mathfrak{S}_0) = \pm 1.$$

On the other hand, let Ω_n be the unimodular group consisting of all integral $n\times n$ matrices with determinant ± 1. Then for $n>2$, Ω_n is its own commutator subgroup [3]. Hence for $n>2$, U_0 is a product of commutators in Ω_n, and therefore $\mathfrak{R}_0$ is in the commutator subgroup of Γ_{2n}. Therefore $\theta(\mathfrak{R}_0)=+1$, so Γ_{2n} has no nontrivial real linear characters for $n>2$.

Now we must prove that there exists a homomorphism of Γ_{2n} into $\{\pm 1\}$ which maps each generator $\mathfrak{R}_0$, $\mathfrak{R}_2$, $\mathfrak{T}_0$, $\mathfrak{S}_0$ into -1, for the cases $n=1$ and $n=2$. This is already known for $n=1$ [3], but we give an independent proof here. Let H be the normal subgroup of Γ_{2n} consisting of all matrices $\equiv I^{(2n)}$ (mod 2). Then it is known [4] that $\Gamma_{2n}/H\cong S_{3n}$ for $n=1, 2$, where S_k is the symmetric group on k symbols. Let π be the homomorphism mapping Γ_{2n} onto S_{3n}, and let A_{3n} be the alternating subgroup of S_{3n}. Then $\pi^{-1}(A_{3n})$ is a subgroup of index 2 of Γ_{2n}, $n=1, 2$. Therefore Γ_{2n} has a nontrivial real linear character for $n=1, 2$, and the previous discussion shows that it is unique, and maps each generator onto -1.

2. Now we consider Γ'_{2n}, and we begin with $n=1$, the most difficult case. The commutator subgroup of $\Gamma_2/\{\pm I\}$ is known [5], but we shall not use this earlier result. According to [6], $\Gamma_2=\{S, T\}$ has as defining relations

$$S^4 = TS^{-1}TS^{-1}TS = 1.$$

Then the sum of the exponents to which S (resp. T) occurs in any relation, must be of the form $4a-b$ (resp. $3b$), where a and b are integers. For $X\in\Gamma_2$ let $\alpha_X=$sum of the exponents to which S occurs, and let $\beta_X=$sum of the exponents to which T occurs, when X is expressed as a power product of S and T. Then $X\in\Gamma'_2$ implies that α_X, β_X are of the form

$$\alpha_X = 4a - b, \qquad \beta_X = 3b, \tag{4}$$

for integral a, b. On the other hand,

$$S^{4a-b}T^{3b} = S^{-b}T^{3b} \equiv (S^{-1}T^3)^b \pmod{\Gamma'_2},$$

and

$$S^{-1}T^3 = S^{-1}T\cdot ST^{-1}ST^{-1}S^{-1}\cdot T \in \Gamma'_2.$$

Hence $X\in\Gamma'_2$ if and only if (4) holds. Consequently $T^{12}\in\Gamma'_2$, $T^m\notin\Gamma'_2$ for $m=1, \cdots, 11$, and we have

$$\Gamma_2 = \bigcup_{m=0}^{11} T^m \Gamma_2'.$$

Thus[3] $(\Gamma_2:\Gamma_2') = 12$.

Next we show how Γ_2' may be defined by means of congruences. Let

$$H_m = \{X \in \Gamma_2 : X \equiv I \pmod{m}\}.$$

Then H_m is a normal subgroup of Γ_2, and in particular Γ_2/H_3 is a group of order 24 consisting of all 2×2 matrices of determinant $+1$ with elements in $GF(3)$. This group contains a normal subgroup $\{C, D\}$ [4] of index 3, where

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Hence the group K_3 consisting of all elements of Γ_2 congruent (mod 3) to a matrix in $\{C, D\}$ is a normal subgroup of Γ_2 of index 3. Therefore $\Gamma_2' \subset K_3$.

Next we remark that Γ_2/H_4 is of order 48, and contains the normal subgroup $\{A, E, F\}$ of order 12 generated by

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

taken mod 4. If K_4 is the set of all elements of Γ_2 congruent mod 4 to a matrix in $\{A, E, F\}$, then K_4 is a normal subgroup of Γ_2 of index 4. Therefore $\Gamma_2' \subset K_4$. Since $K_4K_3 = \Gamma_2$, it follows at once that

$$(\Gamma_2 : K_3 \cap K_4) = 12,$$

and so

$$\Gamma_2' = K_3 \cap K_4.$$[4]

3. We show next that $(\Gamma_4:\Gamma_4') = 2$, and also we determine Γ_4' by means of congruences. From [3] we find that $\mathfrak{R}_0\mathfrak{R}_2$ and $\mathfrak{R}_2^2 \in \Gamma_4'$. Hence $L = \Gamma_4' \cup \mathfrak{R}_2\Gamma_4'$ is a normal subgroup of Γ_4, and (using (3)) $\mathfrak{R}_0$, $\mathfrak{R}_2$, and $\mathfrak{T}_0$ are elements of L. Also we have

$$\mathfrak{S}_0\mathfrak{T}_0 = \mathfrak{T}_0\mathfrak{S}_0\mathfrak{T}_0^{-1}\mathfrak{S}_0^{-1}(\mathfrak{S}_0\mathfrak{T}_0\mathfrak{S}_0^{-1})^2\mathfrak{T}_0^2,$$

so $\mathfrak{S}_0 \in L$. Hence $L = \Gamma_4$, and therefore either $\Gamma_4' = \Gamma_4$ or $(\Gamma_4:\Gamma_4') = 2$.

[3] This result has been obtained independently by Professor J. L. Brenner.

[4] The author wishes to acknowledge with thanks some helpful conversations with Professor E. V. Schenkman on the material in §2.

However we have already seen that Γ_4 contains a subgroup K of index 2. Since $\Gamma_4' \subset K$, we then have $\Gamma_4' = K$.

Finally we remark that the previous discussion shows easily that $\Gamma_{2n}' = \Gamma_{2n}$ for $n > 2$.

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