SOME PROPERTIES OF THE PROPER VALUES OF A MATRIX[1]

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Introduction. In [9], we consider a square matrix $A=(a_{ij})$ with real or complex elements satisfying the inequalities

$$s_j = \sum_{i=1}^{n} |a_{ij}| \leqq 1 \qquad (j = 1, 2, \cdots, n). \tag{1}$$

Then $I-A$ is nonsingular if

$$|a_{11}| < 1, \qquad |a_{kk}| + \sum_{i=1}^{k-1} s_i |a_{ik}| < 1 \qquad (k > 1). \tag{2}$$

With property (1), condition (2) is implied by

$$\sum_{i=1}^{j} |a_{ij}| < 1 \qquad (j = 1, 2, \cdots, n). \tag{3}$$

In [10], we show that if A is non-negative and satisfies (1), then properties (2) and (3) are equivalent, and either property (2) or (3) is a necessary and sufficient condition for the maximal proper value to be less than unity. In this paper, we study some properties of the proper values of a matrix without condition (1). Without assuming (1), property (3) is necessary but not sufficient for a maximal proper value of a non-negative matrix to be less than 1. With condition (1), property (2) is a sufficient condition but not necessary for a real- or complex-valued matrix to have a maximal proper value less than unity in modulus. In §3, we give some equivalent conditions for all the proper values of a non-negative matrix A to be less than unity in modulus. Property (v) in §3 shows that $a_{kk}+c_k<1$ (see (3.4) and (3.5) below) is an equivalent condition for all the proper values to lie within a unit circle. This property is analogous to property (2) above. Another equivalent property is stated as follows: there exists a principal submatrix $A_{(p-1)}$ and a positive vector w (both) of order $n-1$ such that $w'A_{(p-1)}<w'$ and $\det(I-A)>0$.

1. **Moduli of finite matrices.** The modulus of a finite matrix A with real or complex elements is a finite real-valued function, $\|A\|$, satisfying the following axioms:

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(I) $\|sA\| = |s| \cdot \|A\|$ for every number s.
(II) $\|I\| = 1$, independent of the order of I.
(III) $\|AB\| \leqq \|A\| \cdot \|B\|$.
(IV) $\|A+B\| \leqq \|A\| + \|B\|$.
(V) For every submatrix E of an identity matrix I, $\|E\| \leqq \|I\|$.
(VI) If $\lim A_p = A$, then $\lim_p \|A - A_p\| = 0$.
(VII) If A, B are two non-negative matrices, then $\|A+B\| \geqq \max(\|A\|, \|B\|)$.

The first five axioms give the properties of "norms" in Banach algebras. Axiom (VI) is valid only for finite matrices. The assumption of non-negativeness in (VII) is essential. For if A and B are real or complex valued and $B = -A \neq 0$, then (VII) is not valid. Axiom (V) is equivalent to (V_0): The modulus of a matrix is not less than the modulus of any one of its submatrices. Axiom (VII) may be replaced by the following statement: (VII_0) Let the matrices A and B be non-negative such that $A \geqq B$ (i.e. each element of A is at least equal to the corresponding element of B); then $\|A\| \geqq \|B\|$.

The following lemma depends on only (I), (II), (III), and (VI).

LEMMA 1.[2] *The modulus of a maximum proper value of a finite square matrix A with real or complex elements is equal to* $\lim_p \|A^p\|^{1/p}$.

PROOF. The limit stated in the lemma exists and is finite. For, let $a_p = \log \|A^p\|$. Then (II) shows that $a_0 = 0$ and (III) shows that $a_{p+q} \leqq a_p + a_q$. Pólya and Szegö [5, p. 171, Problem 98] show that $\lim (a_p/p) = \inf (a_p/p) = b \geqq -\infty$. Then $\exp(b) \leqq \|A\|$, proving that the limit is finite.

Let λ_1 be a proper value with maximum modulus. Then $\lambda_1^p x = A^p x$, where x is not a zero vector. We may assume that $\|x\| = 1$. Hence by (I) and (III), we have $|\lambda_1|^p \leqq \|A^p\|$, and hence $|\lambda_1| \leqq \|A^p\|^{1/p}$.

Let $r = \lim \|A^p\|^{1/p}$. We shall show that $|\lambda_1|$ cannot be less than r. If $|\lambda_1| < r$, let s be such that $|\lambda_1| < s < r$. Then the Carl Neumann's series $R(s, A) = s^{-1} \sum_0^\infty s^{-p} A^p$ converges. It follows that $s^{-p}A^p$ converges to zero as p tends to infinity. By the continuity property of the modulus function (see Axiom (VI)), for sufficiently large p, $\|s^{-p}A^p\| < 1$, and hence $\|A^p\|^{1/p} < s$ for sufficiently large p. This result is absurd, as $s < r$.

For future development, we mention two instances. Consider real- or complex-valued matrices and vectors. Let A^* denote the conjugate-transpose of A with m rows and n columns. Thus x^*x gives a non-negative number. We introduce

[2] A generalized result in terms of spectral norm in Banach algebra is known.

$$\|x^*\|_1 = \|x\|_1 = \left(\sum_i |x_i|^2\right)^{1/2}; \tag{1.1}$$

$$\|x^*\|_2 = \max(|x_1|, \cdots, |x_m|), \qquad \|x\|_2 = \sum_i |x_i|. \tag{1.2}$$

Define for $\nu = 1, 2$,

$$\|A\|_\nu = \sup\{|x^*Ay| \mid \|x^*\|_\nu = \|y\|_\nu = 1\}. \tag{1.3}$$

Then

$$\|A\|_1 = \|A^*\|_1; \qquad \|A^*A\|_1 = \|AA^*\|_1 = \|A\|_1^2 = \lambda, \tag{1.4}$$

where λ is a maximum proper value of A^*A. See [**4**, pp. 1042–1044];

$$\|A\|_2 = \max_k \left\{\sum_j |a_{jk}|\right\}. \tag{1.5}$$

Let us prove (1.5). We can verify that

$$|x^*Ay| \leqq \|x^*\|_2\|A\|_2\|y\|_2 = \|A\|_2 \tag{1.6}$$

if the moduli of x^* and y are equal to 1. On the other hand, suppose that the rth column gives the maximum value. Let $\bar{a}_{jr}$ be the conjugate of a_{jr}. Take $x^* = (\bar{a}_{jr}/|a_{jr}| \mid j = 1, \cdots, m)$, and $y = \delta_r$, the rth column of the identity matrix of order n. Then $\|x^*\|_2 = \|y\|_2 = 1$ and $|x^*Ay| = \sum_j |a_{jr}|$, which together with (1.6) gives (1.5).

One can verify that (1.4) or (1.5) satisfies Axioms (I) to (VII). To prove the continuity property (VI) of $\|A\|_1$, we make use of $\|A\|_1 \leqq (\sum_{i,j} |a_{ij}|^2)^{1/2}$. If A consists of non-negative numbers, then x, y may be restricted to be non-negative in (1.3) without altering the modulus of A. This fact is used to prove (VII). The results in §1 hold for real quaternions.

2. **Finite square matrices and their proper values.** In the sequel, we shall let λ_1 be a proper value of A with maximum modulus.

THEOREM 1. *Let $A = (a_{ij})$ be a finite square matrix of order n with non-negative elements. If $\lambda_1 < 1$, then by permutations of rows and columns, A has the property*

$$\sum_{i=1}^{k} a_{ik} < 1, \qquad k = 1, 2, \cdots, n. \tag{2.1}$$

A similar property holds for the rows of A.

PROOF. Let $s_k = \sum_{i=1}^n a_{ik}$ for $k = 1, 2, \cdots, n$. The minimum of all s_k is less than 1. For, if $\min_k (s_k) \geqq 1$, then $\lambda_1 \geqq 1$. Thus $\lambda_1 < 1$ implies that there exists at least one column-sum, say s_n, less than unity. From Axiom (VII) or (VII$_0$), one can deduce that if B is a non-negative matrix whose elements are not greater than the corresponding ones of A, then (by Lemma 1), the maximum proper value of B is not greater than that of A. Hence, the maximum proper value of any principal submatrix of A is not greater than that of A. Let A_k be the principal submatrix consisting of the first k rows and columns of A. Then there exists in A_k a column-sum less than unity. Let such a column be the kth one. This proves (2.1).

Note that condition (2.1) is not sufficient for $\lambda_1 < 1$ as counter-examples show. Property (2.1) is not valid if we merely assume A to be real-valued, for, in our proof, Axiom (VII) is used.

If we assume that all s_k are at most unity, then (2.1) is sufficient for $\lambda_1 < 1$. But this condition is even valid for real or complex valued matrices.

THEOREM 2. *Let A be a finite square matrix of order n with real or complex elements such that $s_k = \sum_{i=1}^{n} |a_{ik}| \leqq 1$ for all k. Then*

$$\sum_{i=k}^{n} |a_{ik}| < 1, \qquad k = 1, 2, \cdots, n, \tag{2.2}$$

implies

$$|a_{kk}| + \sum_{j=k+1}^{n} s_j |a_{jk}| < 1, \quad k = 1, \cdots, n-1; \ |a_{nn}| < 1. \tag{2.3}$$

Property (2.3) *implies the existence of an integer $p \leqq n$ such that $\|A_p\|_2 < 1$.*

$\|A\|_2$ is defined by (1.5). The ith row and jth column of A are denoted by $A(i,\cdot)$ and $A(\cdot, j)$ respectively. $A(i, j)$ and a_{ij} have the same meaning.

PROOF. We can see easily that (2.2) implies (2.3). By (2.3), we have $\|A(\cdot, 1)\|_2 < 1$. Let $c_j = \|A^q(\cdot, j)\|_2$, where $0 < q < n$. Suppose that $c_r < 1$ for $r = 1, 2, \cdots, q$. Then for $k = 1, 2, \cdots, q$,

$$\|A^{q+1}(\cdot, k)\|_2 \leqq \|A\|_2 \|A^q(\cdot, k)\|_2 \leqq 1 \cdot c_k < 1. \tag{2.3a}$$

Let $k = q+1$ or larger; then

$$\begin{aligned} \sum_{i=1}^{n} |A^{q+1}(i, k)| &= \sum_{i=1}^{n} |A^q(i,\cdot)A(\cdot, k)| \\ &\leqq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |A^q(i, j)| \cdot |A(j, k)| \right) \\ &= \sum_{r=1}^{q} c_r |a_{rk}| + \sum_{t>q} c_t |a_{tk}|. \end{aligned} \tag{2.4}$$

Now we have two cases. (i) $a_{1k}=\cdots=a_{qk}=0$; then the preceding expression is

$$= \sum_{t>q} c_t |a_{tk}| \leqq \sum_{t=q+2}^{n} s_t |a_{tk}| + |a_{q+1,k}| < 1.$$

(ii) $a_{rk}\neq 0$ for some $r(=1, \cdots, q)$; as $c_r<1$ the last expression in (2.4) is

$$< \sum_{r=1}^{q} |a_{rk}| + \sum_{t>q} c_t |a_{tk}| \leqq \sum_{i=1}^{n} |a_{ik}| = s_k \leqq 1.$$

In either case, we have $\|A^{q+1}(\cdot, k)\|_2<1$ for $k=q+1$ at least. Thus our theorem is proved.

The important fact is the smallness of the integer p. If conditions $s_k\leqq 1$ for all k and (2.3) are not assumed, then the integer p is usually very large, unless A is nilpotent. The preceding theorem is valid for real quaternions.

LEMMA 2. *If there exist an integer p and a non-negative number c such that $\|A^p\|\leqq c<1$, then $\lim_q\|A^q\|^{1/q}<1$ and conversely.*

The lemma is well known in Banach algebras. (See the proof of Lemma 1.)

COROLLARY 1. *If condition (2.2) or (2.3) is satisfied, then $|\lambda_1|<1$.*

LEMMA 3. *Let A be real- or complex-valued and indecomposable such that $s_k\leqq 1$ for $k>1$ and $s_1<1$. Then condition (2.2) holds, subject to the permutations of rows and columns.*

PROOF. Condition (2.2) is satisfied for $k=1$. Let $k=2, \cdots, n$. Consider the submatrix (a_{pq}) for $p=1, \cdots, k-1, q=k, \cdots, n$. The indecomposability shows that there is at least one element different from zero. If one of the elements $a_{1k}, a_{2k}, \cdots, a_{k-1,k}$ is different from zero, then

$$\sum_{i=k}^{n} |a_{ik}| < s_k \leqq 1.$$

If any other element is different from zero, we make the necessary interchange of rows and columns to achieve the desired result. (Lemma 3 holds also for real quaternions.)

By Lemma 3 and Corollary 1, we get the following result of A. T. Brauer [1, pp. 876–877].

COROLLARY 2. *Let A have the properties in the hypothesis of Lemma 3. Then all the proper values of A are less than 1 in modulus.*

3. Non-negative matrices. In this section we give some equivalent conditions for $\lambda_1 < 1$. Not all the conditions are new; for example, (i) is due to Frobenius [2] and (ii) is due to Carl Neumann [3]. However, our proof is elementary and simple. We shall assume A to be non-negative-valued.

LEMMA 4. *If $(I-A)^{-1}$ exists and has non-negative values, then $a_{kk} < 1$ for all k, and the diagonal elements of $(I-A)^{-1}$ are at least equal to unity.*[3]

PROOF. Let (r_{ij}), $i, j = 1, \cdots, n$, be the inverse of $I-A$. The inner product of the kth row of (r_{ij}) and the kth column of $I-A$ gives

$$r_{kk}(1 - a_{kk}) - \sum_{h \neq k} r_{kh} a_{hk} = 1. \tag{3.1}$$

If $1 - a_{kk} \leqq 0$, the left-hand side of (3.1) would be nonpositive, which is impossible. Thus $1 - a_{kk} > 0$. From (3.1), it follows that $r_{kk}(1 - a_{kk}) \geqq 1$. As $0 < 1 - a_{kk} \leqq 1$, we have $r_{kk} \geqq 1$.

The following properties are mutually equivalent.

(i) The inverse of $I-A$ exists and has non-negative values.

(ii) The series $I + A + A^2 + \cdots$ converges.

(iii) Every principal submatrix B of A has the property that the inverse of $I-B$ exists and is non-negative.

Before stating property (iv), we adopt some notations. For p ranging from 1 to n, let A_p be the principal submatrix consisting of the elements in the first p rows and columns of A; A_{-p} the principal submatrix of order $n-p$ omitting all the elements in the first p rows and columns; $C = (a_{ir})$, $D = (a_{sj})$ with $i, j = 1, \cdots, p$ and $r, s = p^{+1}, \cdots, n$. We express $I-A$ in the form

$$I - A = \begin{pmatrix} L & -C \\ -D & M \end{pmatrix} = \begin{pmatrix} I & 0 \\ -DL^{-1} & I \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & K_p \end{pmatrix} \begin{pmatrix} I & -L^{-1}C \\ 0 & I \end{pmatrix} \tag{3.2}$$

where $L = I - A_p$, $M = I - A_{-p}$, and

$$K_p = I - A_{-p} - D(I - A_p)^{-1}C. \tag{3.3}$$

(iv) For each $p = 1, \cdots, n-1$, the inverses of $I - A_p$ and K_p exist and are non-negative.

(v) For each $p = 1, 2, \cdots, n-1$, the inverse of $I - A_p$ exists and is non-negative, and for $k = 1, 2, \cdots, n$,

$$c_k < 1 - a_{kk}, \tag{3.4}$$

where

[3] A related result was obtained by J. H. Curtiss in "*Monte Carlo*" *methods for the iteration of linear operators*, Journal of Mathematics and Physics vol. 32 (1954) p. 224.

$$(3.5)\qquad c_1 = 0, \qquad c_k = \sum_{r,s=1}^{k-1} a_{kr}(I - A_{k-1})^{-1}(r, s)a_{sk}.$$

Similar results hold for every permutation on $1, \cdots, n$.

(vi) There exists a sequence of principal submatrices $A_{(p)}$, $p = 1, \cdots, n$, such that (1) $A_{(p)}$ of order p is a principal submatrix of $A_{(p+1)}$, and (2) $d_{p+1} \leqq d_p$ where $d_p = \det (I - A_{(p)})$. Moreover, $d_n > 0$.

(vii) There exists a sequence of principal submatrices $A_{(p)}$, $p = 1, \cdots, n$, such that (1) $A_{(p)}$ of order p is a principal submatrix of $A_{(p+1)}$, and (2) $\det (I - A_{(p)}) > 0$ for $p = 1, \cdots, n$.

(viii) To each vector y with non-negative components, the equation $x'(I-A) = y'$ has a solution such that $x \geqq y$. Moreover, if $y_k > 0$ for some k, then $x_k > 0$.

(ix) A vector z with positive components exists such that $z'A < z'$.

(x) The maximum proper value λ_1 of A is less than unity.

Proof. We shall show that the first seven properties imply each other in cyclic order. (i) *implies* (ii): By Lemma 4, we write $(I-A)^{-1} = I + A_*$; then A_* has non-negative values and commutes with A. For $m \geqq 1$,

$$A_* = A + A^2 + \cdots + A^m + A^m A_*.$$

Let $S_m = I + A + \cdots + A^m$. Then $S_m \leqq S_{m+1}$ and every element of S_m is bounded above by the corresponding element of $(I-A)^{-1}$. Hence $\lim_m S_m$ exists and is the inverse of $I - A$. That (ii) *implies* (iii) is obvious. (iii) *implies* (iv): From (iii), it follows that the inverse of $I - A_p$ exists and is non-negative. That the inverse of K_p exists and is non-negative follows from (3.2) and (iii). (iv) *implies* (v): Let $L = I - A_p$ for $p = 1, 2, \cdots, n-1$ in (3.2). Then (iv) states that the inverse of $I - A_p$ exists and is non-negative. Obviously (3.4) holds for $k = 1$. Let $p = 2, \cdots, n$, and apply (3.2) to $I - A_p$ with $L = I - A_{p-1}$. Then K_p becomes $1 - a_{pp} - c_p$. Since the inverse of $I - A_p$ exists and is non-negative, it follows that $1 - a_{pp} - c_p \neq 0$ and is non-negative, i.e. positive. (v) *implies* (vi): Let $A_{(p)} = A_p$ as specified above. Then property (1) in (vi) is satisfied. We can show [8, p. 234] that for $p = 2, \cdots, n$, $d_p = d_{p-1}(1 - a_{pp} - c_p)$, which by (v), proves property (2) in (vi) and also $d_n > 0$. That (vi) *implies* (vii) is evident. (vii) *implies* (i): Since $d_n > 0$, the inverse of $I - A$ exists. To prove the non-negativeness of $(I-A)^{-1}$, we reduce $I - A$ into a diagonal matrix by a method similar to (3.2). Let $e_1 = d_1$, $e_p = d_p/d_{p-1}$ for $p > 1$. By hypothesis, $e_1, \cdots, e_n$ are positive. Put $B_1 = A$. From the non-negative matrix B_p of order $n - p + 1$ such that the first element in the diagonal of $I - B_p$ is e_p, we construct B_{p+1} of order $n - p$ by the method of (3.2) as follows: Let B_p^0 be the principal submatrix of B_p with the first row and column omitted; β_p and α_p' be the first *column* and *row* respectively of B_p with the first element omitted. Define $B_{p+1}=B_p^0+\beta_p e_p^{-1}\alpha_p'$. Then B_{p+1} is non-negative, and

$$I - B_p = \begin{pmatrix} 1 & 0 \\ -\beta_p e_p^{-1} & I \end{pmatrix} \begin{pmatrix} e_p & 0 \\ 0 & I - B_{p+1} \end{pmatrix} \begin{pmatrix} 1 & -e_p^{-1}\alpha_p' \\ 0 & I \end{pmatrix}.$$

By the elementary properties of determinants, the first element in the diagonal of $I-B_{p+1}$ is e_{p+1}. Thus, $I-A=(I-P)E(I-Q)$, where E is a diagonal matrix with $e_1, \cdots, e_n$ in its diagonal, P and Q have non-negative elements respectively *below* and *above* the principal diagonal, and zeros elsewhere. Hence $(I-A)^{-1}=(I+Q)E^{-1}(I+P)$, which is non-negative. This completes the proof that the first 7 properties are equivalent to one another.

To complete the proof, we shall show that (i), (viii), (ix), (x), and (iii) imply one another in that order. That (i) *implies* (viii) follows from Lemma 4. (viii) *implies* (ix): For a positive y, we have, from (viii), a positive u such that $u'-u'A=y'>0$. (ix) *implies* (x): If A is nilpotent, then $\lambda_1=0<1$. If A is not nilpotent, by Frobenius' result [2], $\lambda_1>0$ and $Av=\lambda_1 v$ where v is non-negative. Hence $\lambda_1 u'v =u'Av<u'v$. Since $u'v>0$, we have property (x). That property (x) *implies* (iii) is a well known result concerning the Carl Neumann's series [7, pp. 18–19].

Note that the weak condition (vii) implies that all the principal minors of $I-A$ are positive. The following equivalent condition is useful in practical applications:

COROLLARY. *The maximal proper value λ_1 of A is less than* 1 *if and only if there exist a principal submatrix $A_{(n-1)}$ and a positive vector w (both) of order $n-1$ such that $w'A_{(n-1)}<w'$ and* $\det(I-A)>0$.

Added in the proof. There is redundance in (v). The property given by (3.4) and (3.5) implies the existence of non-negative inverses of $I-A_p$ for all p. The proof is by induction. See the demonstration for which (vii) *implies* (i).

REFERENCES

1. A. T. Brauer, *Limits for the characteristic roots of a matrix*, III, Duke Math. J. vol. 15 (1948) pp. 871–877.

2. G. Frobenius, *Über Matrizen aus nicht-negativen Elementen*, K. Preuss. Akad. Sitzungsber. (1912) pp. 456–477.

3. C. Neumann, *Untersuchungen über das logarithmische und Newtonsche Potential*, Leipzig, Teubner, 1877, XVI, p. 368.

4. J. von Neumann and H. Goldstine, *Inverting matrices of higher order*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099.

5. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, I, Berlin, Springer, 1925.

6. G. B. Price, *Bounds for determinants with dominant principal diagonal*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 497–502.

7. A. Wintner, *Spektraltheorie der unendlichen Matrizen*, von S. Hirzel, 1929.

8. Y. K. Wong, *Some inequalities of determinants of Minkowski type*, Duke Math. J. vol. 19 (1952) pp. 231–241.

9. ———, *An inequality for Minkowski matrices*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 139–141.

10. ———, *On nonnegative valued matrices*, Proc. Nat. Acad. Sci. U.S.A. vol. 40 (1954) pp. 121–124.

11. J. H. Curtiss, *"Monte Carlo" methods for the iteration of linear operators*, Journal of Mathematics and Physics vol. 32 (1954) pp. 209–232.

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PROJECTIONS IN THE SPACE (m)[1]

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A *projection* in a Banach space is a continuous linear mapping P of the space into itself which is such that $P^2=P$. Two closed linear manifolds M and N of a Banach space B are said to be *complementary* if each z of B is uniquely representable as $x+y$, where x is in M and y in N. This is equivalent to the existence of a projection for which M and N are the range and null space [7, p. 138]. It is therefore also true that closed linear subsets M and N of B are complementary if and only if the linear span of M and N is dense in B and there is a number $\epsilon>0$ such that $\|x+y\|\geqq\epsilon\|x\|$ if x is in M and y in N.

It is known that a Banach space M is complemented in each Banach space in which it can be embedded if it is isomorphic with a complemented subspace of the space (m) of bounded sequences. In particular, if M is a subspace of a Banach space Z and is isometric with a subspace M' of (m), then there is a projection of Z onto M of norm less than or equal to λ if there is a projection of (m) onto M' of norm equal to λ (see [8, p. 538] and [9, p. 945]). Thus the existence of a complement in (m) for a subspace M of (m) is independent of the method by which M is embedded in (m). Any separable Banach space is isometric with a subspace of (m) [3, p. 107]. Hence a separable Banach space is complemented in each space in which it can be em-

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