

THE EMBEDDING OF HOMEOMORPHISMS IN FLOWS

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1. Introduction. Let X be a topological space, and let R be the real number system. If F is a function on $X \times R$, $x \in X$ and $t \in R$, we will usually denote $F(x, t)$ by $F_t(x)$. For each real number t , F_t denotes the obvious function on X .

A (topological) flow on X is a continuous function F on $X \times R$ into X such that:

- (1) F_t is a homeomorphism of X onto X for each $t \in R$; and
- (2) $F_t(F_s(x)) = F_{t+s}(x)$ for all $t \in R$, $s \in R$ and $x \in X$.

We now state the general embedding problem for flows.

EMBEDDING PROBLEM. For a given space X and a given homeomorphism f of X onto X , does there exist a flow F on X for which $F_1 = f$?

If such a flow F exists, we say that f is embedded in F .

In general, the embedding problem is quite difficult. In this paper we discuss only the case in which X is an interval of real numbers.

If X is an interval of real numbers and f is a continuously differentiable homeomorphism of X onto X , we may ask whether or not f can be embedded in a flow F for which each homeomorphism F_t has a continuous derivative. We obtain some results pertaining to the solution of this latter problem, although a complete solution is not obtained.

2. The embedding problem for intervals. Let f be a homeomorphism of an interval of real numbers onto itself. In order that it be possible to embed f in a flow, it is obviously necessary that f be order preserving. We prove that this condition is also sufficient.

LEMMA 1. If h is a homeomorphism of a closed interval $[a, b]$ onto itself such that a and b are the only fixed points of h , then it is possible to embed h in a flow H such that if $a < x < b$ and $-1 < t < 1$ then $H_t(x)$ is between $h^{-1}(x)$ and $h(x)$.

PROOF. It is proved in [1] and [2] that there exists an order preserving homeomorphism ψ on $[a, b]$ onto $[0, \infty]$ and a positive number A such that $h(x) = \psi^{-1}(A\psi(x))$ for $a \leq x \leq b$. If $f(x) > x$ for $a < x < b$,

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then $A > 1$; if $f(x) < x$ for $a < x < b$, then $A < 1$. If we now define $H_t(x) = \psi^{-1}(A^t \psi(x))$ for $a \leq x \leq b$ and $t \in R$, then it is easy to verify that H has the desired properties.

THEOREM 1. *If f is an order preserving homeomorphism of an interval J onto itself, then it is possible to embed f in a flow F .*

PROOF. We may assume without loss of generality that J is a closed interval. Let K be the set of all fixed points of f , and let Σ be the set of all closed intervals which are the closures of the components of $J - K$. For each interval $u \in \Sigma$, we use Lemma 1 to obtain a flow U on u such that $U_1 = f|u$, and such that for $-1 < t < 1$ and each point x that is interior to U , $u_t(x)$ is between $f^{-1}(x)$ and $f(x)$.

Now let x be a member of J . If $x \in K$, we define $F_t(x) = x$ for all $t \in R$. If $x \in J - K$, then there exists $u \in \Sigma$ such that $x \in u$; and in this case we define $F_t(x) = U_t(x)$ for all $t \in R$.

For each t , F_t is a one-to-one order preserving function on J onto J , and it follows that F_t is a homeomorphism. Moreover, if $t \in R$, $s \in R$ and $x \in J$, then it is easy to verify that $F_t(F_s(x)) = F_{t+s}(x)$.

We must finally prove that F is continuous on $J \times R$. Since each F_t is continuous and $F_t F_s = F_{t+s}$ for all t, s , it is sufficient to prove that F is continuous at each point of the form $(a, 0)$, $a \in J$. If $a \in J - K$, it is obvious that F is continuous at $(a, 0)$. Thus, let us assume that $a \in K$ and that $(x_n, t_n) \rightarrow (a, 0)$ as $n \rightarrow \infty$. For all large values of n we have $-1 < t_n < 1$, and hence $F(x_n, t_n)$ is between $f^{-1}(x_n)$ and $f(x_n)$. Since f and f^{-1} are continuous and $f(a) = f^{-1}(a) = F(a, 0)$, it follows that $F(x_n, t_n) \rightarrow F(a, 0)$ as $n \rightarrow \infty$. Thus F is continuous.

If the function h of the lemma has a continuous derivative, then the function ψ may be chosen so as to have a continuous derivative on the open interval (a, b) . It follows that each function H_t has a continuous derivative on the open interval (a, b) . Turning now to the above theorem, we see that if f has a continuous derivative on J , then the flow F may be constructed so that each F_t has a continuous derivative on $J - K$. We have no assurance, however, that the functions F_t will have derivatives at fixed points of f , even though f has a derivative at these points. An interesting problem is that of determining conditions for f that will guarantee that we can construct the flow F in such a manner that each function F_t has a continuous derivative on all of J . We obtain a few results pertaining to this problem in the next section.

3. Flows of continuously differentiable functions. Throughout this section we assume that f is a function which has the following properties:

- (i) f is a homeomorphism of a half open interval $(a, b]$ onto itself;
- (ii) f has a continuous derivative on $(a, b]$;
- (iii) $f(x) > x$ for $a < x < b$;
- (iv) $f'(x) > 0$ for $a < x \leq b$;
- (v) f' is monotone nonincreasing on the interval $(a, b]$.

We are going to prove that there exists a unique flow F such that $F_1 = f$ and such that F_t has a continuous derivative on $(a, b]$ for each real number t . It is clear that if such a flow exists, then f must commute with F_t for each t . Thus, in trying to construct the flow F , it is reasonable to first try to determine the set of all continuously differentiable order preserving homeomorphisms that commute with f .

Using conditions (i)–(v) above, it is possible to prove the following lemma. Since the proof is fairly straightforward, it is omitted.

LEMMA 2. *If $a < x < b$ and $a < y < b$, then the infinite product*

$$\prod_{n=0}^{\infty} [f'(f^n(x))/f'(f^n(y))]$$

converges. Moreover, if $a < a^ < b^* < b$, then for each fixed y , the infinite product converges uniformly in x for $a^* \leq x \leq b^*$.*

In view of the above lemma, we obtain a continuous function ϕ if we define $c = (a+b)/2$ and then define $\phi(x) = \prod_{n=0}^{\infty} [f'(f^n(x))/f'(f^n(c))]$ for $a < x < b$.

THEOREM 2. *If g is a continuously differentiable homeomorphism of $(a, b]$ onto itself and g commutes with f , then*

$$g'(x) = g'(b)\phi(x)/\phi(g(x))$$

for $a < x < b$.

PROOF. Since f and g commute, we obtain for all x ,

$$f(g(x)) = g(f(x)).$$

We now differentiate each side of the above equation, obtaining

$$f'(g(x))g'(x) = g'(f(x))f'(x).$$

We solve for $g'(x)$, obtaining

$$g'(x) = [f'(x)/f'(g(x))]g'(f(x)).$$

Next we replace x by $f^k(x)$ in the above equation, and use the fact that g and f^k commute. We obtain

$$g'(f^k(x)) = [f'(f^k(x))/f'(f^k(g(x)))]g'(f^{k+1}(x)).$$

If n is a positive integer, it follows that

$$g'(x) = \left\{ \prod_{k=0}^n \left[\frac{f'(f^k(x))}{f'(f^k(g(x)))} \right] \right\} g'(f^{n+1}(x)).$$

Since g' is continuous at b and $\lim_{n \rightarrow \infty} f^{n+1}(x) = b$, we obtain

$$g'(x) = \left\{ \prod_{k=0}^{\infty} \left[\frac{f'(f^k(x))}{f'(f^k(g(x)))} \right] \right\} g'(b) = g'(b) \frac{\phi(x)}{\phi(g(x))}.$$

COROLLARY 1. *Under the same hypotheses as for the above theorem, $g'(b) \neq 0$.*

It is easy to prove under the hypotheses for f that $0 < f'(b) < 1$. We define $A = f'(b)$. It follows from the above results that any continuously differentiable homeomorphism on $(a, b]$ that commutes with f must be a solution of a differential equation of the form

$$dy/dx = A^t \phi(x)/\phi(y).$$

We now study the existence and uniqueness of solutions of the differential equation

$$E(t): dy/dx = A^t \phi(x)/\phi(y).$$

LEMMA 3. *Suppose that t is a real number and that $a \leq d < b$. Then there exists at most one continuous function g on $(d, b]$ such that g satisfies $E(t)$ on (d, b) and $g(b) = b$.*

PROOF. Suppose that g and h are continuous on $(d, b]$, both are solutions of $E(t)$ on the interval (d, b) , and $g(b) = h(b) = b$. If g and h are not identical, then there exists a point p in the interval such that $g(p) \neq h(p)$. We may assume without loss of generality that $g(p) > h(p)$. There exists a point q , $p < q \leq b$, such that $g(q) = h(q)$ and $g(x) > h(x)$ for $p \leq x < q$. We define $w(x) = g(x) - h(x)$. The Mean Value Theorem yields a point r , $p < r < q$, for which

$$\begin{aligned} \frac{g(p) - h(p)}{p - q} &= \frac{w(p) - w(q)}{p - q} = w'(r) = g'(r) - h'(r) \\ &= A^t \phi(r) \left\{ \frac{1}{\phi(g(r))} - \frac{1}{\phi(h(r))} \right\}. \end{aligned}$$

Since $g(r) > h(r)$ and ϕ is nonincreasing, it follows that $A^t \phi(r) \{1/\phi(g(r)) - 1/\phi(h(r))\}$ is nonnegative. We now have a contradiction, since $[g(p) - h(p)]/(p - q)$ is negative.

We next define a function Φ on the interval $(a, b]$ by letting $\Phi(x) = \int_x^b \phi(t) dt$.

LEMMA 4. Φ is an order reversing homeomorphism of $(a, b]$ onto $[0, \infty)$.

PROOF. It is obvious that Φ is continuous and strictly monotone decreasing, and that $\Phi(b) = 0$. Hence it is sufficient to prove that $\lim_{x \rightarrow a} \Phi(x) = \infty$.

Let us assume that $a < x < c$. We first observe that

$$\Phi(x) \geq \int_x^c \phi(t) dt.$$

However, for $x \leq t \leq c$, $\phi(t) \geq \prod_{k=0}^n [f'(f^k(t))/f'(f^k(c))]$ for each n . Moreover, it is easy to prove by induction on n that

$$\prod_{k=0}^n f'(f^k(t)) = \frac{d}{dt} f^{n+1}(t).$$

It follows from these facts that $\phi(x) \geq [f^{n+1}(c) - f^{n+1}(x)] / \prod_{k=0}^n f'(f^k(c))$ for every n . Now let B be any positive number. It is easy to see that $\lim_{n \rightarrow \infty} f^{n+1}(c) = b$ and $\lim_{n \rightarrow \infty} \prod_{k=0}^n f'(f^k(c)) = 0$. Thus there exists an integer m such that $f^{m+1}(c) > (a + 2b)/3$ and

$$\prod_{k=0}^m f'(f^k(c)) < (b - a)/3B.$$

Now choose $\epsilon > 0$ such that if $a < x < a + \epsilon$ then $f^{m+1}(x) < (2a + b)/3$. It now follows that if $a < x < a + \epsilon$ then $\Phi(x) > B$. Therefore,

$$\lim_{x \rightarrow a} \Phi(x) = \infty.$$

LEMMA 5. If g is an order preserving homeomorphism of $(a, b]$ onto itself and g commutes with f , then $\lim_{x \rightarrow b} \phi(x)/\phi(g(x)) = 1$.

PROOF. It is easy to verify that $\lim_{x \rightarrow b} \phi(x)/\phi(f^m(x)) = 1$ if m is an integer. We shall show that there exists an integer n such that $f^{n-1}(x) \leq g(x) \leq f^{n+2}(x)$ for $a < x < b$. Since ϕ is monotone, our lemma will follow from the inequalities

$$\phi(x)/\phi(f^{n-1}(x)) \leq \phi(x)/\phi(g(x)) \leq \phi(x)/\phi(f^{n+2}(x)).$$

There exists an integer n such that $f^n(c) \leq g(c) \leq f^{n+1}(c)$. Now let x be any member of the interval (a, b) . There exists an integer k such that $f^k(c) \leq x \leq f^{k+1}(c)$. If we now use the fact that each power of f is monotone increasing and commutes with g , we obtain

$$f^{n-1}(x) \leq f^{n+k}(c) \leq g(f^k(c)) \leq g(x) \leq g(f^{k+1}(c)) \leq f^{n+k+2}(c) \leq f^{n+2}(x).$$

Thus, $f^{n-1}(x) \leq g(x) \leq f^{n+2}(x)$, and our lemma follows.

THEOREM 3. *There exists a unique flow F on $(a, b]$ such that $F_1 = f$ and such that each F_t is continuously differentiable on $(a, b]$. Moreover, if g is any continuously differentiable homeomorphism of $(a, b]$ onto itself such that g commutes with f , then there exists a real number t such that $F_t = g$.*

PROOF. We define $F_t(x) = \Phi^{-1}(A^t \Phi(x))$ for $a < x \leq b$. It is easy to verify that each function F_t is a homeomorphism of $(a, b]$ onto itself. Moreover, it is easy to prove that $F_s F_t = F_{s+t}$ for all s and t , and that F is continuous. Thus F is a flow on the interval $(a, b]$.

Since $\Phi'(x) = -\phi(x)$, it follows that $F'_t(x) = A^t \phi(x) / \phi(F_t(x))$ and hence F_t satisfies the differential equation $E(t)$ on the interval (a, b) . Since f obviously satisfies $E(1)$, it follows from Lemma 3 that $F_1 = f$.

Since $F_1 = f$ and F is a flow, it follows that F_t commutes with f for each t . We now use Lemma 5 to obtain $\lim_{x \rightarrow b} F'_t(x) = A^t$. It follows that F_t is differentiable at b , and that $F'_t(b) = A^t$. Thus each function F_t has a continuous derivative on $(a, b]$.

Let us now prove that the flow F is unique. We observe that if n is a positive integer and g is a continuously differentiable homeomorphism of $(a, b]$ onto itself for which $g^n = f$, then

$$f'(x) = g'(g^{n-1}(x))g'(g^{n-2}(x)) \cdots g'(x).$$

If we let x tend to b , it is easy to see that we obtain

$$f'(b) = (g'(b))^n.$$

It follows that g satisfies the equation $E(1/n)$. Now let G be any flow of continuously differentiable functions which satisfies $G_1 = f$. We see that $G_{1/n}$ and $F_{1/n}$ both satisfy $E(1/n)$ for each positive integer n and hence $G_{1/n} = F_{1/n}$. It follows that $G_r = F_r$ for every rational number r . Since flows are continuous in both variables, $G_t(x) = F_t(x)$ for all real t and all x in the interval $(a, b]$. Therefore F is unique.

Now suppose that g is a continuously differentiable homeomorphism of $(a, b]$ onto itself and that g commutes with f . It follows from Corollary 1 that $g'(b) \neq 0$. Thus, since g is order preserving, $g'(b) > 0$ and there exists a real number t such that $A^t = g'(b)$. It follows from Theorem 2 that g satisfies the differential equation $E(t)$, and since F_t also satisfies $E(t)$, Lemma 3 implies that $F_t = g$.

It is obvious that we may extend the domain of f and of the functions F_t to the closed interval $[a, b]$ by defining $F_t(a) = a$, and we shall then obtain a flow F on $[a, b]$. However, as is shown by the following example, it is not necessarily true that each of the functions

F_t will have a derivative at a , even though f has a derivative at a .

EXAMPLE. Let $g(x) = 4x/(3x+1)$ for $0 \leq x \leq 1$. Next define a function f such that $f(x) = 2x$ for $0 \leq x \leq 1/3$; $f(x) = g(x)$ for $1/2 \leq x \leq 1$; $f(x)$ arbitrary for $1/3 \leq x \leq 1/2$, subject only to the requirement that f satisfy the conditions (i)–(v) listed at the beginning of §3. It is easily seen that it is possible to define such a function f . Theorem 3 implies that there exist flows G and F such that $G_1 = g$, $F_1 = f$ and, all of the functions F_t and G_t are continuously differentiable on $(0, 1]$. If one makes use of the uniqueness of G , it is easily verified that $G_t(x) = 4^t x / ((4^t - 1)x + 1)$. Since $g(x) = f(x)$ for $1/2 \leq x \leq 1$, it is easily seen that $G_{1/2}$ and $F_{1/2}$ satisfy the same differential equation on the interval $1/2 \leq x \leq 1$. It follows from Lemma 3 that $F_{1/2}(x) = G_{1/2}(x)$ for $1/2 \leq x \leq 1$, and we use this fact to compute $F_{1/2}(1/2) = 2/3$. Next we define points p_0, p_1, p_2, \dots by letting $p_0 = 1/2$, $p_1 = 1/3$, and $p_n = F_{-1}(p_{n-2})$ for $n \geq 2$. Since $F_{1/2}(p_0) = 2/3 = F_1(p_1)$, we obtain $F_{1/2}(p_1) = p_0$. It is easy to prove by induction that $F_{1/2}(p_n) = p_{n-1}$ for every positive integer n . Since $F_{-1}(x) = x/2$ for $0 \leq x \leq 1/2$, it is easy to compute the values of the numbers p_n , and to verify that the difference quotient $(F_{1/2}(p_n) - F_{1/2}(0))/(p_n - 0)$ is equal to $3/2$ if n is odd and is equal to $4/3$ if n is even. Thus $F_{1/2}$ does not have a derivative at 0.

This example demonstrates that it is possible to have a function f which is continuously differentiable on the closed interval $[a, b]$ and which satisfies conditions (i)–(v) on the half-open interval $(a, b]$, but which cannot be embedded in a flow of functions that are continuously differentiable on the closed interval $[a, b]$.

We do, however, have the following result.

THEOREM 4. *If f is continuously differentiable over the closed interval $[a, b]$ and satisfies (i)–(v) on $(a, b]$, then the set G of all continuously differentiable order preserving homeomorphisms g of $[a, b]$ onto itself for which g commutes with f forms a group which is isomorphic to either the group of real numbers or the group of integers.*

PROOF. We use Theorem 3 to obtain a flow F on $[a, b]$ such that each of the functions F_t has a continuous derivative on $(a, b]$, and $F_1 = f$. It follows from Theorem 3 that if $g \in G$, then there exists a real number t such that $F_t = g$. We let S be the set of all real numbers t for which $F_t \in G$. If we use the fact that $F'_{t+s}(x) = F'_t(F_s(x))F'_s(x)$ for $a < x < b$, it is easy to see that if t and s are members of S then $t+s$ is also a member of S . Moreover, if t is in S then $-t$ is in S . Thus S is a subgroup of the reals. It is not difficult to prove that G is a group which is isomorphic to S , and consequently our theorem is

proved if we can show that either S is the set R of all real numbers or else S is a nondegenerate discrete subgroup of R .

We observe that S is nondegenerate since $1 \in S$. Now assume that S is isomorphic to neither R nor the group of integers. It then follows that both S and $R-S$ are dense in R , and this implies that $R-S$ is uncountable. We shall now show that this is impossible. In order to accomplish this, we define u_t to be the lower derivate of F_t at a , and we define v_t to be the upper derivate of F_t at a . It is easily seen that if $s < t$, then $F_s(x) < F_t(x)$ for $a < x < b$, and it follows that if $s < t$ then $u_s \leq u_t$ and $v_s \leq v_t$. Now let r and t be members of $R-S$, $r < t$. Since S is dense in R , there exists $s \in S$ such that $r < s < t$. We see that $u_r < v_r \leq v_s = u_s \leq u_t < v_t$. Thus, by defining $L(t)$ to be the open interval (u_t, v_t) for each $t \in R-S$, we obtain a one-to-one function L that maps $R-S$ onto a set of mutually exclusive open intervals. This implies that $R-S$ is countable, and we have a contradiction.

We conclude by listing two unsolved problems that are of interest.

PROBLEM 1. *Find conditions on f that are necessary and sufficient for $S=R$, where S and R are the sets defined in the proof of Theorem 4.*

PROBLEM 2. *Replace condition (v) on f by a weaker condition.*

The solution of these problems would constitute a major step toward obtaining necessary and sufficient conditions that it be possible to embed a continuously differentiable homeomorphism (having arbitrarily many fixed points) of an interval onto itself in a flow of continuously differentiable homeomorphisms.

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