A NOTE ON NONSINGULAR FORMS IN A FINITE FIELD

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1. Let $q = p^n$, $p > 2$, r a fixed integer ≥ 1 . The writer has shown [1, Theorem 8.3] that one can construct quadratic forms $Q_1, \dots, Q_r$

$$(1) \quad Q_h(x) = \sum_1^r \alpha_{hi} x_i x_j \quad (\alpha_{hi} \in GF(q)),$$

such that

$$(2) \quad \det (\xi_1 Q_1 + \dots + \xi_r Q_r) \neq 0$$

for arbitrary $(\xi_1, \dots, \xi_r) \neq (0, \dots, 0)$, $\xi_i \in GF(q)$. This result suggests the possibility of finding r homogeneous forms $f_1, \dots, f_r$ of degree m , where $(q, m) = 1$ such that $f_1, \dots, f_r$ vanish simultaneously only at $(0, \dots, 0)$, and secondly

$$(3) \quad f = \xi_1 f_1 + \dots + \xi_r f_r \quad (\xi_i \in GF(q))$$

has no singular point (except at $(0, \dots, 0)$) for arbitrary ξ_i not all zero.

To construct such forms let β be a number of $GF(q^r)$ such that $\beta, \beta^q, \dots, \beta^{q^{r-1}}$ are linearly independent relative to $GF(q)$; Hensel first proved the existence of such β . Now put

$$(4) \quad \phi_i(x) = \sum_{j=1}^r \beta^{q^{i+j}} x_j^m \quad (i = 1, \dots, r).$$

Let $\gamma_1, \dots, \gamma_r$ be numbers of $GF(q^r)$ that are linearly independent relative to $GF(q)$ and put

$$(5) \quad x_j = \sum_{k=1}^r \gamma_k^{q^j} y_k \quad (j = 1, \dots, r).$$

Define $f_i(y)$ by means of

$$(6) \quad f_i(y) = \phi_i(x) \quad (i = 1, \dots, r).$$

Then in the first place

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$$\begin{aligned}
 f_i^q(y) &= \phi_i^q(x) = \sum_j \beta^{q^{i+j+1}} \left(\sum_k \gamma_k^{q^{i+1}} y_k^q \right)^m \\
 &= \sum_j \beta^{q^{i+j}} \left(\sum_k \gamma_k^{q^j} y_k^q \right)^m \\
 &= f_i(y^q),
 \end{aligned}$$

so that the coefficients of $f_i(y)$ are in $GF(q)$.

Now assume that

$$(7) \quad f_1(y) = \cdots = f_r(y) = 0$$

for some $(y_1, \dots, y_r)$. Then by (4) and (6), (7) implies

$$(8) \quad \sum_{j=1}^r \beta^{q^{i+j}} x_j^m = 0 \quad (i = 1, \dots, r).$$

But since $\beta, \beta^q, \dots, \beta^{q^{r-1}}$ are linearly independent we have that the determinant

$$(9) \quad \det(\beta^{q^{i+j}}) \neq 0.$$

Consequently (8) implies $x_1 = \cdots = x_r = 0$ and therefore (7) holds only for $y_1 = \cdots = y_r = 0$.

Consider next the form $f(y)$ defined by (3). We have

$$(10) \quad \frac{\partial f(y)}{\partial y_i} = \sum_{j,k} \xi_j \frac{\partial \phi_j(x)}{\partial x_k} \frac{\partial x_k}{\partial y_i} = m \sum_{j,k} \xi_j \beta^{q^{j+k}} x_k^{m-1} \gamma_i^{q^k}.$$

We assume that

$$(11) \quad \frac{\partial f(y)}{\partial y_i} = 0 \quad (i = 1, \dots, r)$$

for some $(y_1, \dots, y_r) \neq (0, \dots, 0)$. If we put

$$\eta_k = \sum_j \xi_j \beta^{q^{j+k}},$$

then (10) and (11) imply

$$(12) \quad \sum_k \eta_k x_k^{m-1} \gamma_i^{q^k} = 0 \quad (i = 1, \dots, r).$$

But the linear independence of $\gamma_1, \dots, \gamma_r$ is equivalent to the non-vanishing of the determinant $\det(\gamma_i^{q^k})$; thus (12) implies

$$\eta_k x_k^{m-1} = 0 \quad (k = 1, \dots, r).$$

Since not all x_k vanish it follows that

$$(13) \quad \sum_i \xi_i \beta^{q^i+k} = 0$$

for at least one value of k . But since $\xi_i \in GF(q)$, raising (13) to the q th power it follows that (13) holds for all $k=1, \dots, r$. But in view of (9) this implies $\xi_1 = \dots = \xi_r = 0$.

We have therefore proved the following

THEOREM 1. *Let $(q, m)=1$, $r \geq 1$. There exist homogeneous polynomials $f_1, \dots, f_r$ of degree m with coefficients in $GF(q)$, that vanish simultaneously only at $(0, \dots, 0)$ and such that*

$$f = \xi_1 f_1 + \dots + \xi_r f_r \quad (\xi_i \in GF(q))$$

has no singular point (except at $(0, \dots, 0)$).

The condition $\xi_i \in GF(q)$ is evidently essential.

2. Returning to the case of quadratic forms, the result in (2) cannot be improved. For given $r+1$ quadratic forms, then

$$(14) \quad \det (\xi_1 Q_1 + \dots + \xi_{r+1} Q_{r+1})$$

is a polynomial of degree r in the ξ . Consequently by a well known theorem of Chevalley [2], the determinant (14) vanishes for some $(\xi_1, \dots, \xi_{r+1}) \neq (0, \dots, 0)$.

For the case of arbitrary forms of degree m let us take

$$(15) \quad f(y) = \xi_1 f_1(y) + \dots + \xi_s f_s(y) \quad (\xi_i \in GF(q))$$

and consider the Hessian

$$(16) \quad H_f = \det (\partial^2 f / \partial y_i \partial y_j) \quad (i, j = 1, \dots, r)$$

as a polynomial in the ξ 's, H_f is of degree $r(m-2)$. Hence if

$$(17) \quad s > r(m-2)$$

Chevalley's theorem applies. We may state

THEOREM 2. *Let $f_1(y), \dots, f_s(y)$ be arbitrary homogeneous polynomials of degree m with coefficients in $GF(q)$ and let (17) hold. Then for arbitrary $y_i \in GF(q)$ there exist $\xi_i \in GF(q)$ such that the Hessian H_f vanishes at $(y_1, \dots, y_r)$.*

REFERENCES

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2. C. Chevalley, *Démonstration d'une hypothèse de M. Artin*, Abh. Math. Sem. Hansischen Univ. vol. 11 (1936) pp. 73-75.