The asymptotic representation of a solution $u_{j,k}(x, \lambda)$ given by the theorem does not in general hold over all of R_x but only on the image of $\Xi_{j,k}$.

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ON STIELTJES INTEGRATION

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Among the theorems concerning the Stieltjes integral there are two which are established for integrals in one-dimensional space, but not in spaces of more than one dimension. These are (I) if $\int f dg$ exists, f and g have no common discontinuity; (II) if $\int f dg$ exists, and g is of bounded variation and t is its total-variation function, then $\int f dt$ exists. The method of proof for one dimension[1] does not extend to higher dimensions. In this note extensions of these theorems to n dimensions are proved for the ordinary Stieltjes integral and for a modified form of it.[2]

1. **Definitions.** Throughout this note we shall assume that f is real-valued and bounded on a set D in the space R^n, and that g is real-valued on R^n. For each interval $I \subset R^n$ we define $\Delta_g I$ in the usual way, as the sum of 2^n terms each of which is ± 1 times the value of g at a vertex of I. If B is a closed interval contained in D, an *extended partition* of B is a set $P = \{I_1, I_2, \cdots, I_k, x_1, x_2, \cdots, x_k\}$ in

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[1] L. M. Graves, *Theory of functions of real variables*, McGraw-Hill, 1946, p. 263, Theorem 4, and p. 273, Theorem 14.

[2] E. J. McShane and T. A. Botts, *A modified Riemann-Stieltjes integral*, Duke Math. J. vol. 19 (1952) p. 293.

which the I_j are nonoverlapping closed intervals whose union contains B, and for each j the point x_j is in $D \cap I_j$. P is a *restricted partition* of B if it is an extended partition and $\bigcup I_j = B$. The *mesh* of P is the greatest of the diameters of the I_j in P. For each extended partition P, define

$$S(P) = \sum_{j=1}^{k} f(x_j)\Delta_g(I_j \cap B).$$

If this approaches a limit as the mesh of P approaches 0, the limit is the *modified Stieltjes integral* of f with respect to g over B, and we denote it simply by $\int_B f(x)dg(x)$. If $S(P)$ has a limit as mesh $P \to 0$ subject to the condition that P is a restricted partition, the limit is the (ordinary) Stieltjes integral of f with respect to g over B.

Whenever the modified integral exists so does the ordinary integral, and the two are equal (the reason for defining the modified integral is that it possesses some desirable properties which the ordinary integral lacks). But it is easy to see that if $f = (f(x): x \in D)$, the ordinary Stieltjes integral of f with respect to g over B is identical with the modified Stieltjes integral of its restriction $f_B = (f(x): x \in B)$ with respect to g over B.

2. **Two lemmas.** If $\int_B f(x)dg(x)$ exists, to each $\epsilon > 0$ corresponds $\delta(\epsilon) > 0$ such that if mesh $P < \delta(\epsilon)$, then $|S(P) - \int_B f(x)dg(x)| < \epsilon$. Let us define $O(I)$ to be the oscillation of f on $I \cap D$ if this is nonempty, and to be 0 if $I \cap D$ is empty. Since in any finite set of nonoverlapping closed intervals of diameter $< \delta$ those which meet D can be included in a partition of mesh $< \delta$, we readily establish the following lemma.

LEMMA 1. *If the* (*modified Stieltjes*) *integral* $\int_B f(x)dg(x)$ *exists, and* $(\delta(\epsilon): \epsilon > 0)$ *is defined as above, and* $\epsilon > 0$, *and* $I_1, \cdots, I_k$ *are nonoverlapping closed intervals of diameter* $< \delta(\epsilon/2)$, *then*

$$\sum_{j=1}^{k} O(I_j) \left| \Delta_g(I_j \cap B) \right| < \epsilon.$$

In order to save verbosity, by a *hyperplane* we shall always mean a set $\{x: x^{(i)} = C\}$, where i is one of the numbers $1, \cdots, n$ and C is real. We define a sequence $C_1, C_2, \cdots$ of integers recursively by the relations

$$C_0 = 2, C_n = 2 + C_0 + \binom{n}{1}C_1 + \binom{n}{2}C_2 + \cdots + \binom{n}{n-1}C_{n-1} \qquad (n = 1, 2, 3, \cdots).$$

LEMMA 2. *Let* ϵ *be a positive number, and let* I *be a closed interval such that for each closed interval* $I' \subset I$, $O(I')|\Delta_g(I' \cap B)| < \epsilon$. *Let* I *be* *subdivided by k ($k \leqq n$) mutally perpendicular hyperplanes into non-overlapping intervals $I_1, I_2, \cdots, I_{2^k}$ whose intersection contains a point x_0. Then*

$$O(I)\,|\Delta_g(I_j \cap B)| < C_k\epsilon \quad (j = 1, 2, \cdots, 2^k).$$

For brevity we write $G(I_j)$ for $\Delta_g(I_j \cap B)$.

Without loss of generality we may assume that x_0 is the origin and that the k cutting hyperplanes are those defined by the k equations $x^{(i)}=0$ $(i=1, \cdots, k)$. For each I_j, let the "signature" $\sigma(I_j)$ be the set of integers i in $\{1, 2, \cdots, k\}$ for which the ith coordinate of the center of I_j is negative. If $\sigma=\sigma(I_j)$, we can use $I(\sigma)$ as another name for I_j. The number of elements in σ will be denoted by $|\sigma|$.

It is easily seen that because x_0 is in each I_j, the oscillation of f on some I_j must be at least $O(I)/2$. By reversing some axes if necessary we can bring it about that the interval $I(\phi)$, whose signature is the empty set, has this property.

We now establish inductively

(*) If $1 \leqq j \leqq 2^k$, and $\sigma=\sigma(I_j)$, then

$$O(I)\,|G(I_j)| < C_{|\sigma|}\epsilon.$$

First suppose σ empty, so that $I_j=I(\phi)$. Then $O(I_j) \geqq O(I)/2$, so

$$O(I)\,|G(I_j)| \leqq 2O(I_j)\,|G(I_j)| < 2\epsilon = C_0\epsilon.$$

Next suppose statement (*) true for $|\sigma(I_j)| < h$; we prove it true if $|\sigma(I_j)| = h$. There are 2^h intervals I_m in the set $\{I_1, \cdots, I_{2^k}\}$ with $\sigma(I_m) \subset \sigma(I_j)$; for simplicity we may assume the notation chosen so that these are $I(\phi)=I_1, I_2, \cdots, I_{2^h}=I_j$. The union of these intervals is a closed interval I^*, and

$$G(I_j) = G(I^*) - \sum_{i=1}^{j-1} G(I_i).$$

Also $O(I^*) \geqq O(I_1) \geqq O(I)/2$. Letting $\sum'$ denote the sum over all proper subsets of $\sigma(I_j)$, we have

$$\begin{aligned} O(I)\,|G(I_j)| &\leqq O(I)\,|G(I^*)| + \sum_{i=1}^{j-1} O(I)\,|G(I_i)| \\ &\leqq 2O(I^*)\,|G(I^*)| + \sum{}' C_{|\sigma|}\epsilon \\ &< 2\epsilon + \sum_{i=0}^{h-1} \binom{h}{i} C_i\epsilon \\ &= C_h\epsilon. \end{aligned}$$

So (*) holds for $|\sigma(I_j)| = h$, and by induction holds for $0 \leqq |\sigma(I_j)| \leqq k$. Since $C_0 \leqq C_1 \leqq \cdots \leqq C_k$, the lemma is established.

3. A theorem on discontinuities. The interval function Δ_g is continuous at a point x if to each $\epsilon > 0$ corresponds $\delta > 0$ such that whenever I is a closed interval of diameter less than δ and having $x \in I$, $|\Delta_g I| < \epsilon$.

THEOREM I. *If the modified Stieltjes integral $\int_B f(x)dg(x)$ exists, and x_0 is a point of B at which $f = (f(x): x \in D)$ is discontinuous, then the interval function $(\Delta_g(I \cap B): I$ an interval$)$ is continuous at x_0.*

Let ϵ be the oscillation of f at x_0, and let γ be positive. By Lemma 1, there exists $\delta > 0$ such that for every closed interval I^* of diameter less than 2δ, $O(I^*)|\Delta_g(I^* \cap B)| < \gamma\epsilon$. Let I be a closed interval of diameter less than δ with $x_0 \in I$; let V be the vertex of I farthest from x_0, and C the vertex of I farthest from V. Define I^* to be the closed interval with center C and a vertex at V; its diameter is twice that of I, hence less than 2δ. Also, except in the trivial case of degenerate I, x_0 is interior to I^*, so $O(I^*) \geqq \epsilon$. The n hyperplanes through C divide I^* into 2^n nonoverlapping closed intervals having C in common, and I is one of these. By Lemma 2,

$$\epsilon \left| \Delta_g(I \cap B) \right| \leqq O(I^*) \left| \Delta_g(I \cap B) \right| < C_n \gamma \epsilon,$$

so $|\Delta_g(I \cap B)| < C_n\gamma$. Since γ is an arbitrary positive number this completes the proof.

COROLLARY. *If this (ordinary) Stieltjes integral $\int_B f(x)dg(x)$ exists, there is no point of B at which the interval-function $(\Delta_g(I \cap B): I$ an interval$)$ and the restriction of f to B, $f_B = (f(x): x \in B)$ are both discontinuous.*

4. Interval-functions of bounded variation. If Δ_g is of bounded variation over B, there exists a function t in R^n such that for each closed interval $I \subset B$, $\Delta_t I$ is the total variation of Δ_g over I.

THEOREM II. *Let f be a bounded function on a domain D in R^n, g a function on R^n such that Δ_g is of bounded variation on a closed interval $B \subset D$, and t a function on R^n such that $\Delta_t I$ is the total variation of Δ_g over I for each closed interval $I \subset B$. If the (modified) Stieltjes integral $\int_B f(x)dg(x)$ exists, so does $\int_B f(x)dt(x)$.*

Write $G(I)$ for $\Delta_g(I \cap B)$ and $T(I)$ for $\Delta_t(I \cap B)$, and let M be the upper bound for $|f(x)|$ on D. Let $(\delta(\epsilon): \epsilon > 0)$ be defined as before Lemma 1, and let ϵ be a positive number. There exists a finite set of nonoverlapping closed intervals $J_1, \cdots, J_q$ whose union is B such that

$$\sum_{i=1}^{q} |G(J_i)| > T(B) - \epsilon.$$

Without loss of generality we may suppose that the J_i are obtained by cutting B by hyperplanes, and have diameter less than $\delta(\epsilon/2)$.

Let δ' be a positive number less than the least of the edges of the intervals $J_1, \cdots, J_q$. We now investigate an extended partition $P = \{I_1, \cdots, I_m, x_1, \cdots, x_m\}$ of mesh less than δ'. For each I_h, the nonempty intervals in the set $I_h \cap J_1, \cdots, I_h \cap J_q$ are obtained by cutting I_h by hyperplanes, and have a common point x_h'. If we define

$$c_h = \sup \{O(I') |G(I')| : I' \subset I_h\},$$

by Lemma 1 we have $c_1 + \cdots + c_m \leqq \epsilon$. By Lemma 2

$$O(I_h) |G(I_h \cap J_i)| \leqq C_n c_h \qquad (i = 1, \cdots, q),$$

and at most 2^n intervals J_i have points in common with I_h, so

$$\sum_i O(I_h) |G(I_h \cap J_i)| \leqq 2^n C_n c_h.$$

Then

$$\begin{aligned} \sum_h O(I_h) T(I_h) &= \sum_{i,h} O(I_h) T(I_h \cap J_i) \\ &\leqq \sum_h 2^n C_n c_h + \sum_{i,h} 2M \{T(I_h \cap J_i) - |G(I_h \cap J_i)|\} \\ &< (2^n C_n + 2M)\epsilon. \end{aligned}$$

Again by Lemma 1,

$$\sum_i O(J_i) |G(J_i)| < \epsilon,$$

whence

$$\begin{aligned} \sum_i O(J_i) T(J_i) &\leqq \sum_i O(J_i) |G(J_i)| + 2M \sum_i \{T(J_i) - |G(J_i)|\} \\ &< (2M + 1)\epsilon. \end{aligned}$$

Let ξ_i be the center of J_i. Both x_h and x_h' are in I_h, so $|f(x_h) - f(x_h')| \leqq O(I_h)$; and unless $I_h \cap J_i$ is empty both ξ_i and x_h' are in J_i, whence $|f(\xi_i) - f(x_h')| \leqq O(J_i)$. Hence

$$\left| \sum_h f(x_h)T(I_h) - \sum_i f(\xi_i)T(J_i) \right|$$
$$\leqq \left| \sum_h [f(x_h) - f(x_h')]T(I_h) \right| + \left| \sum_{i,h} [f(x_h') - f(\xi_i)]T(I_h \cap J_i) \right|$$
$$\leqq \sum_h O(I_h)T(I_h) + \sum_{i,h} O(J_i)T(I_h \cap J_i)$$
$$< (2^n C_n + 4M + 1)\epsilon.$$

If $P^* = \{I_1^*, \cdots, I_l^*, x_1^*, \cdots, x_l^*\}$ is any extended partition also of mesh less than δ', the same argument applies to it as to P, so

$$\left| \sum_{h=1}^{m} f(x_h)T(I_h) - \sum_{j=1}^{l} f(\overset{*}{x_j})T(\overset{*}{I_j}) \right| < 2(2^n C_n + 4M + 1)\epsilon.$$

By the Cauchy criterion the limit of the sum $\sum f(x_h)T(I_h)$ exists as mesh P tends to zero, and by definition this limit is $\int_B f(x)dt(x)$.

COROLLARY. *If Δ_g is of bounded variation on B, and t is a function such that $\Delta_t I$ is the total variation of Δ_g on I for each closed interval $I \subset B$, and f is bounded and the (ordinary) Stieltjes integral $\int_B f(x)dg(x)$ exists, so does $\int_B f(x)dt(x)$.*

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