A TAUBERIAN THEOREM FOR α -CONVERGENCE OF CESÀRO MEANS

A. V. BOYD

The object of this note is to generalize certain Tauberian results proved by Gehring [3] for summability $(C, k; \alpha)$. The notation is as in [3], with the following additional definitions: If $k > -1$, then A_n^k, B_n^k denote the n th Cesàro sums of order k for the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, where $b_n = na_n$. A_n^{-1}, B_n^{-1} denote a_n, b_n . Summability $(C, -1; \alpha)$ of $\sum a_n$ will be taken to mean summability $(C, 0; \alpha)$ of $\sum a_n$ together with the condition that sequence $\{na_n\}$ be α -convergent to 0. Gehring's Tauberian theorems are:

THEOREM 4.3.2. *Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is summable $(A; \alpha)$ to S . If the sequence $\{na_n\}$ is α -convergent to 0, $\sum a_n$ is α -convergent to S .*

THEOREM 4.3.3. *Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is summable $(A; \alpha)$ to S . Then $\sum a_n$ is α -convergent to S if and only if the sequence $\{(a_1 + \dots + na_n)/n\}$ is α -convergent to 0.*

THEOREM 4.3.4. *Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is α -convergent. If the sequence $\{na_n\}$ is α -convergent to 0, $\sum a_n$ is summable $(C, k; \alpha)$ to its sum for every $k > -1$.*

These will be used in the proof of the following:

THEOREM. *Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is summable $(A; \alpha)$ to S . Then, for $r \geq -1$, $\sum a_n$ is summable $(C, r; \alpha)$ to S if and only if the sequence $\{B_n^r/C_n^{r+1}\}$ is α -convergent to 0.*

PROOF OF THEOREM. *Necessity.* If $r = -1$ this follows immediately from the definition of summability $(C, -1; \alpha)$. If $r > -1$ then by the consistency theorem for $(C, r; \alpha)$ summability (Gehring [3, Theorem 4.2.1]) it follows that both sequences $\{S_n^r\}, \{S_n^{r+1}\}$ are α -convergent to S . By Hardy [1, Equation (6.1.6)],

$$(1) \quad S_n^r = S_n^{r+1} + \frac{1}{r+1} \frac{B_n^r}{C_n^{r+1}}$$

and the result follows since a linear combination of sequences summable $(C, k; \alpha)$ is itself summable $(C, k; \alpha)$.

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Sufficiency. If $r > -1$ it may be shown¹ as in Szász [4, §1], that

$$(2) \quad \frac{1}{y+1} \sum_{n=0}^{\infty} S_n^{r+1} \left(1 - \frac{1}{y+1}\right)^n = \frac{r+1}{y} \int_0^y \left(1 - \frac{u}{y}\right)^r \phi(u) du,$$

where $\phi(u) = f(1 - 1/(u+1))$ and $f(x) = \sum a_n x^n$.

CASE (i). $\alpha = 0, r > -1$. Since $\sum a_n$ is summable (A) to S it follows that $\phi(u)$ tends to S as u increases. The right-hand side of (2), being the $(r+1)$ th transform of $\phi(u)$, also tends to S ; and so the sequence $\{S_n^{r+1}\}$ is summable $(A; \alpha)$ to S .

CASE (ii). $0 < \alpha \leq 1, r > -1$. Putting

$$g(y) = \frac{1}{y+1} \sum_{n=0}^{\infty} S_n^{r+1} \left(1 - \frac{1}{y+1}\right)^n,$$

we get, from (2), that

$$g(y) = (r+1) \int_0^1 (1-v)^r \phi(vy) dv,$$

where $\phi(u)$ now has bounded α -variation over $(0, \infty)$. Let

$$\begin{aligned} V &= \left[\sum_{\nu=1}^N |g(y_\nu) - g(y_{\nu-1})|^{1/\alpha} \right]^\alpha \\ &= (r+1) \left[\sum_{\nu=1}^N \left| \int_0^1 (1-v)^r \{ \phi(vy_\nu) - \phi(vy_{\nu-1}) \} dv \right|^{1/\alpha} \right]^\alpha. \end{aligned}$$

Then by Theorem 201 of [5],²

$$\begin{aligned} V &\leq (r+1) \int_0^1 (1-v)^r \left\{ \sum_{\nu=1}^N | \phi(vy_\nu) - \phi(vy_{\nu-1}) |^{1/\alpha} \right\}^\alpha dv \\ &\leq (r+1) M \int_0^1 (1-v)^r dv \\ &= M, \end{aligned}$$

where $M = V_\alpha \{ \phi(x); 0 \leq x < \infty \}$. Thus $g(y)$ has bounded α -variation over $(0, \infty)$ and so the series $\sum s_n$, where $s_n = S_n^{r+1} - S_{n-1}^{r+1}$, is summable $(A; \alpha)$ to S . Further, by Hardy [1, Equation (6.1.6)],

$$n s_n = n(S_n^{r+1} - S_{n-1}^{r+1}) = B_n^r / C_n^{r+1},$$

¹ Note however the error in Szász's equation (2.4). There, and in the previous line, occurs an extraneous term $(1+1/y)^{\alpha-1}$.

² I am indebted to a referee for shortening my argument at this step.

so that sequence $\{ns_n\}$ is α -convergent to 0. Hence by Theorem 4.3.2 we have that $\sum s_n$ and sequence $\{S_n^{+1}\}$ are α -convergent to S .

It is readily seen from Minkowski's inequality that the sum of two α -convergent sequences is also α -convergent, and we therefore deduce from (1) that $\{S_n\}$ is α -convergent to S ; i.e., $\sum a_n$ is summable $(C, r; \alpha)$ to S .

CASE (iii). $r = -1$. When $\alpha = 0$ the result reduces to Tauber's original theorem; when $0 < \alpha < 1$ it follows from Theorem 4.3.2. For $\alpha = 1$ the result was proved by Hyslop [2, Theorem 4].

REFERENCES

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UNIVERSITY OF THE WITWATERSRAND