ASYMPTOTIC SOLUTION WITH RESPECT TO A PARAMETER OF A DIFFERENTIAL EQUATION HAVING AN IRREGULAR SINGULAR POINT

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1. **Introduction.** The subject of this note is the asymptotic solution, for large absolute values of a complex parameter λ, of the differential equation of the type

$$(1)\qquad (z-z_0)^\nu \frac{d^2w}{dz^2} + \lambda(z-z_0)^{\nu/2}P_1(z,\lambda)\frac{dw}{dz} + \lambda^2 P_2(z,\lambda)w = 0,$$

with $\nu-2$ a positive real constant. The complex variable z is to be confined to a bounded neighborhood of the irregular singular point z_0. The analogous problem for a class of nth order differential equations, containing (1) for integral $\nu \geqq 6$, has been discussed by Hurd [**2**]. His analysis is somewhat complicated, and it appears that a clearer and more direct investigation of the asymptotic solutions of (1) than is attainable by the methods applied in [**2**] is possible. This is accomplished largely by adapting the formal structure of the analysis [**3**] used in the discussion of (1) for $\nu=2$.

The case $\nu>2$ discussed here resembles, in some respects, that when $\nu<2$. In order to contrast the case $\nu<2$ with the present one, we first outline the nature of our results.

Suitable changes of the variables in (1) reduce it to the form

$$(2)\qquad L(u) = 0, \qquad L = d^2/dx^2 - \lambda^2 Q(x,\lambda).$$

By way of hypothesis it is to be assumed that the function $x^\nu Q(x,\lambda)$ is single-valued and analytic in x throughout a bounded region R containing the origin and in λ at infinity. The important restriction is made that $x^\nu Q(x,\infty)$ is nonzero in R with λ chosen so that $x^\nu Q(x,\infty)|_{x=0}=1$. A further hypothesis upon R is stated in §3.

Let n be any non-negative integer. It will be shown that a function $\xi(x,\lambda)$ may be defined and a pair of polynomials in $1/\lambda$,

$$(3)\qquad A_{\pm n}(x,\lambda) = \sum_0^n \frac{\alpha_{\pm j}(x)}{\lambda^j},$$

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determined such that the expressions

$$y_{\pm n}(x, \lambda) = e^{\pm\xi} A_{\pm n}(x, \lambda) \tag{4}$$

approximate solutions of (2) with an error of $e^{\pm\xi}O(\lambda^{-n-1})$.[2] The functions $\alpha_{\pm j}(x)$ are determined by means of a simple algorithm so that the results given here are more explicit than those of [2]. This algorithm is essentially the same as that used by Birkhoff in 1908 [1] in the treatment of (1) for $\nu = 0$. The decisive difference is in the choice of $\xi(x, \lambda)$. Moreover, if one were to determine the asymptotic representations of the solutions of (2) in accordance with the procedures developed by Birkhoff [1] and Langer [4], among others, for $\nu < 2$ or by the analysis referred to in [3], the remainder terms multiplying $e^{\pm\xi}$ would not remain bounded as $z \to z_0$. As is the case when $\nu = 0$ or 2, the asymptotic forms given here are made up of elementary functions. As is not the case when $\nu = 0$ or 2 but as is the case in general when $\nu < 2$, the Stokes' phenomenon appears; that is, no solution is asymptotic to one of the expressions $y_{\pm n}(x, \lambda)$ everywhere in R. Previously, asymptotic forms approximating solutions of (2) to terms of order $e^{\pm\xi}O(\lambda^{-n-1})$ where n is any non-negative integer have been found only for $\nu = -2, -1, 0, 1, 2$. In this respect the theory to be given for $\nu > 2$ is more complete than that known for $\nu < 2$. But the fact that z_0 is an irregular singular point of (2) has prevented the author from obtaining connection formulas for the solutions. Such formulas are known if $\nu < 2$ and are not necessary if $\nu = 0$ or 2. The problem of finding connection formulas in λ for fixed z has remained unsolved in all cases.

2. **Definition of** $\xi(x, \lambda)$. The hypotheses upon $Q(x, \lambda)$ guarantee that it may be expanded in the series[3]

$$Q(x, \lambda) = \sum_{0}^{\infty} \frac{q_j(x)}{\lambda^j}, \qquad \text{when } x \neq 0 \text{ and } |\lambda| > N,$$

where the functions $x^\nu q_j(x)$ are analytic in R. This series may be differentiated termwise with respect to x.

Let $\phi(x)$ be the square root of $q_0(x)$ such that

$$x^{\nu/2}\phi(x)\big|_{x=0} = 1. \tag{5}$$

A sequence of polynomials $\kappa_j^*(x)$, $j = 0, 1, \cdots$, of degree $[1 + \nu/2]$ or less is determined by the conditions that the following limits exist

[2] The notation $O(\lambda^{-n})$ is used to designate a bounded function of x which when multiplied by λ^n is a bounded function of λ as well.

[3] The letter N is to be used as a generic symbol for a positive constant.

(if $2<\nu<4$, then $\kappa_2^*(x)$ will be the sum of a polynomial and a term $cx^{\nu-2}$):

$$(6)\quad \begin{aligned} &\text{(a)}\quad \lim_{x\to 0} x^{\nu/2}[\kappa_j^*(x)q_0(x) - q_j(x)] = l_j, \qquad j \neq 0, 2,\\ &\text{(b)}\quad \kappa_0^*(x) \equiv 1,\\ &\text{(c)}\quad \lim_{x\to 0} x^{\nu/2}\left[\kappa_2^*(x)q_0(x) - q_2(x) + \frac{\nu}{4}\left(\frac{\nu}{4} - 1\right)x^{-2}\right] = l_2, \end{aligned}$$

wherein the l_j are constants. Because of the analyticity of $x^\nu Q(x, \lambda)$ in x and λ, it is readily seen that the function $\kappa^2(x, \lambda) - \kappa_2^*(x)/\lambda^2$, where

$$\kappa^2(x, \lambda) = \sum_0^\infty \frac{\kappa_j^*(x)}{\lambda^j},$$

is also analytic for x in R and $|\lambda| > N$. The root

$$\kappa(x, \lambda) = \sum_0^\infty \frac{\kappa_j(x)}{\lambda^j}$$

is chosen so that $\kappa_0(x) \equiv 1$. We now define $\xi(x, \lambda)$ to be any primitive of

$$\xi'(x, \lambda) = \lambda\kappa(x, \lambda)\phi(x). \tag{7}$$

Lastly, we introduce a function $\Theta(x, \lambda)$ through the relation

$$[\xi'(x, \lambda)]^2 - \lambda^2 Q(x, \lambda) = \lambda\Theta(x, \lambda) + \frac{\nu}{4}\left(1 - \frac{\nu}{4}\right)x^{-2}.$$

Referring to the definitions (6) of $\kappa_j^*(x)$, it is evident that $x^{\nu/2}\Theta(x, \lambda)$ is bounded for $|\lambda| < N$ and x in R. Moreover,

$$\Theta(x, \lambda) = \sum_0^\infty \frac{\theta_j(x)}{\lambda^j}, \qquad x \neq 0.$$

3. **The surfaces R_x and R_ξ.** For fixed λ the function $\xi(x, \lambda)$ is in general multiple-valued in R. We therefore consider R to be covered by a Riemann surface appropriate to a single-valued representation of $\xi(x, \lambda)$. This surface will be designated R_x. The Riemann surface over which the inverse function $\xi^{-1}(x, \lambda)$ is single-valued we denote R_ξ. The portions of the positive and negative axes of reals included on R_ξ may be thought of as dividing this surface into regions $\Xi_{j,k}$, $j=1, 2$, $k=0, \pm 1, \pm 2, \cdots$, where

$$(8)\quad \begin{aligned} \Xi_{1,k}&: \quad 2k\pi \leqq \arg(\xi) \leqq 2(k+1)\pi,\\ \Xi_{2,k}&: \quad (2k-1)\pi \leqq \arg(\xi) \leqq (2k+1)\pi. \end{aligned}$$

The boundaries of the images of these regions upon R_x are, of course, dependent upon the parameter λ. If the constant ν is rational, the surfaces R_x and R_ξ may be of finite order; and in this case, only a finite number of the regions $\Xi_{j,k}$ will be distinct. On the other hand, if ν is irrational, the regions $\Xi_{j,k}$ are distinct for all allowed j and k.

A curve joining a point of $\Xi_{j,k}$ to the point at infinity as approached along the ray $\arg(\xi)=(2k-j+2)\pi$ and along which $\Re(\xi)$ is monotonic will be called a Γ-curve. The image of such a curve upon R_x will be called a γ-curve. The final assumption to be made upon the character of the region R may now be stated. We assume that for the value of λ under consideration and for each integer k all points of $\Xi_{j,k}$, $j=1$ or 2, may be connected to $\xi=\infty$ by Γ-curves chosen so that

$$\left|\int_\gamma dx\right| < N, \quad \text{when} \quad \xi(x,\lambda) \text{ is in } \Xi_{j,k}.$$

We note that inasmuch as ξ is dependent upon λ, a single region R may not satisfy the above hypothesis for all λ with $|\lambda|>N$.

4. **Determination of $A_{\pm n}(x,\lambda)$.** Direct computation shows that

$$L(y_{\pm n}) = e^{\pm\xi}H_{\pm n}$$

where

$$\begin{aligned} H_{\pm n}(x,\lambda) = {} & A''_{\pm n}(x,\lambda) \pm 2\xi'(x,\lambda)A'_{\pm n}(x,\lambda) \\ & + \left[(\xi'(x,\lambda))^2 - \lambda^2 Q(x,\lambda) \pm \xi''(x,\lambda)\right]A_{\pm n}(x,\lambda). \end{aligned}$$

The function $\kappa(x,\lambda)$ has been so chosen that it is possible to determine the $\alpha_{\pm j}(x)$ to be bounded functions by equating the first $n+1$ terms of the series in descending powers of λ for $H_{\pm n}(x,\lambda)$ to zero. These terms vanish provided that the α's satisfy the following differential equations

$$\pm 2\phi\alpha'_{\pm 0} + (\theta_0 \pm \phi')\alpha_{\pm 0} = 0,$$

$$\begin{aligned} \pm 2\phi\alpha'_{\pm 1} + (\theta_0 \pm \phi')\alpha_{\pm 1} + \alpha''_{\pm 0} + (\nu/4)(1-\nu/4)x^{-2} & \\ + 2\kappa_1\phi\alpha'_{\pm 0} + \left[\theta_1 \pm (\kappa_1\phi)'\right]\alpha_{\pm 0} &= 0, \end{aligned}$$

$$\begin{aligned} \pm 2\phi\alpha'_{\pm m} + (\theta_0 \pm \phi')\alpha_{\pm m} + \alpha''_{\pm(m-1)} \pm 2\phi \sum_1^m \kappa_j \alpha'_{\pm(m-j)} & \\ + \sum_1^m \left[\theta_j \pm (\kappa_j\,\phi)'\right]\alpha_{\pm(m-j)} &= 0, \\ m = 2, 3, \cdots, n. & \end{aligned}$$

This is the case if

$$\alpha_{\pm 0}(x) = \phi^{-1/2} \exp\left(\mp \int_0^x \frac{\theta_0}{2\phi} dt \right),$$

$$\alpha_{\pm 1}(x) = \mp\, \alpha_{\pm 0}(x) \int_0^x \left\{ \pm \left[\kappa_1 \theta_0 + \kappa_1' \phi \right] \right.$$

$$(9) \qquad \left. + \frac{\nu}{4}\left(1 - \frac{\nu}{4}\right) x^{-2} + \theta_1 + \frac{\alpha''_{\pm 0}}{\alpha_{\pm 0}} \right\} \frac{dt}{2\phi},$$

$$\alpha_{\pm m}(x) = -\, \alpha_{\pm 0}(x) \int_0^x \left\{ 2\phi \sum_1^m \kappa_j \alpha'_{\pm (m-j)} \right.$$

$$\left. + \sum_1^m \left[(\kappa_j \phi)' \pm \theta_j \right] \alpha_{\pm (m-j)} \pm \alpha''_{\pm (m-1)} \right\} \frac{dt}{2\phi \alpha_{\pm 0}}$$

$$m = 2, 3, \cdots, n.$$

The functions $x^{-\nu/4}\alpha_{\pm 0}$ are analytic in R, and the functions $x^{-\nu/4-1}\alpha_{\pm m}$, $m=1, \cdots, n$, are bounded in R. Consequently, $x^{\nu/4}H_{\pm n}(x, \lambda)$ is bounded in R.

The linear independence of $y_{+n}(x, \lambda)$ and $y_{-n}(x, \lambda)$ can now be established. The Wronskian of these functions will be denoted $W(x, \lambda)$, and it is easily seen from (4) that

$$W(x, \lambda) = \begin{vmatrix} A_{+n} & A'_{+n} + \xi' A_{+n} \\ A_{-n} & A'_{-n} - \xi' A_{-n} \end{vmatrix}.$$

At $x=0$ this relation becomes $W(0, \lambda) = -2\lambda\kappa(0, \lambda)$. It can be shown that [3, §3]

$$W'(x, \lambda) = (1/\lambda^n)\left\{ A_{+n}H_{-n} - A_{-n}H_{+n} \right\}.$$

Integrating this expression from zero to z, we may conclude that

$$W(x, \lambda) = -\, 2\lambda\kappa(0, \lambda) + B(x, \lambda)/\lambda^n$$

for x in R and $|\lambda| > N$, where $B(x, \lambda)$ designates a bounded function of x and λ. Thus for $|\lambda|$ sufficiently large $W(x, \lambda)$ is nonzero in R, and therefore the functions $y_{\pm n}(x, \lambda)$ are linearly independent.

5. **Asymptotic solutions of** $L(u)=0$. In order to show the asymptotic representation of solutions of $L(u)=0$ by the functions $y_{\pm n}(x, \lambda)$, we compare $L(u)=0$ with the differential equation satisfied by the functions

$$(10) \qquad z_{\pm n}(x, \lambda) = \left[\lambda / W(x, \lambda) \right]^{1/2} y_{\pm n}(x, \lambda).$$

This equation may be written in the form [3, §3]

$$d^2z/dx^2 - [\lambda^2 Q(x, \lambda) - \Omega(x, \lambda)/\lambda^n]z = 0,$$

in which

$$\Omega(x, \lambda) = -\lambda^n[E/W + (F/2W)^2 + (F/2W)'],$$

$$F(x, \lambda) = \begin{vmatrix} A_{+n} & H_{+n} \\ A_{-n} & H_{-n} \end{vmatrix}, \quad \text{and} \quad E(x, \lambda) = \begin{vmatrix} H_{+n} & A'_{+n} + \xi' A_{+n} \\ H_{-n} & A'_{-n} - \xi' A_{-n} \end{vmatrix}.$$

The observations made in §4 upon the structure at $x=0$ and behavior in R of the functions $\alpha_{\pm j}(x)$, $H_{\pm n}(x, \lambda)$ and $W(x, \lambda)$ reveal that $x^{\nu/2}\Omega(x, \lambda)$ is bounded in R. That $\Omega(x, \lambda)$ is bounded for $|\lambda| > N$ and each fixed $x \neq 0$ may be concluded from the behavior of these same functions at $\lambda = \infty$.

The relation

$$\begin{vmatrix} z_{+n} & z'_{+n} \\ z_{-n} & z'_{-n} \end{vmatrix} = \frac{\lambda}{W} \begin{vmatrix} y_{+n} & y'_{+n} \\ y_{-n} & y'_{-n} \end{vmatrix}$$

follows directly from (10). Consequently, the Wronskian of $z_{+n}(x, \lambda)$ and $z_{-n}(x, \lambda)$ is equal to λ. Thus for each integer k a pair of linearly independent solutions, $u_{1,k}(x, \lambda)$ and $u_{2,k}(x, \lambda)$, of $L(u)=0$ is given by the formulas

$$\begin{aligned} u_{j,k}(x, \lambda) &= z_{\pm n}(x, \lambda) \\ &\quad - \frac{1}{\lambda^{n+1}} \int_\gamma [z_{+n}(x)z_{-n}(t) - z_{+n}(t)z_{-n}(x)]\Omega(t)u_{j,k}(t)dt, \\ &\qquad j = 1, 2,\ k = 0, \pm 1, \cdots. \end{aligned} \tag{11}$$

In these and subsequent formulas the upper sign is to be used when $j=1$, the lower sign when $j=2$. As indicated the path of integration is to be a γ-curve.

With the aid of the abbreviations

$$\begin{aligned} Z_{\pm n} &= \phi^{1/2}e^{\mp\xi}z_{\pm n}, \qquad U_{j,k} = \phi^{1/2}e^{\mp\xi}u_{j,k}, \\ K_j(x, t, \lambda) &= \pm \frac{\Omega(t)}{\phi(t)} [Z_{\pm n}(x)Z_{\mp n}(t) - Z_{\pm n}(t)Z_{\mp n}(x) \\ &\qquad \cdot \exp(\mp 2[\xi(x) - \xi(t)])], \end{aligned} \tag{12}$$

equation (11) may be written as

$$U_{j,k}(x) = Z_{\pm n}(x) - \frac{1}{\lambda^{n+1}} \int_\gamma K_j(x, t, \lambda)U_{j,k}(t)dt. \tag{13}$$

The functions $\Omega(x, \lambda)/\phi(x)$ and $Z_{\pm n}(x, \lambda)$ are bounded on R_x so that the kernel $K_j(x, t, \lambda)$ is a bounded function aside from the exponential factor. The γ-curves have been so chosen that this exponential is bounded along a γ-curve. Thus $K_j(x, t, \lambda)$ is bounded for ξ in $\Xi_{j,k}$.

Equation (13) is formally satisfied by the infinite series

$$U_{j,k}(x) = Z_{\pm n}(x) + \frac{1}{\lambda^n} \sum_{m=1}^{\infty} \frac{Z_{\pm n}^{(m)}(x)}{\lambda^m}, \tag{14}$$

wherein

$$Z_{\pm n}^{(m)}(x) = \int_\gamma K_j(x, t, \lambda) Z_{\pm n}^{(m-1)}(t)dt, \qquad Z_{\pm n}^{(0)}(x) = Z_{\pm n}(x).$$

Let M denote an upper bound of

$$|K_j(x, t, \lambda)| \cdot \left| \int_\gamma dt \right|, \qquad |Z_{\pm n}(x)| \qquad \text{for } \xi(x, \lambda) \text{ in } \Xi_{j,k};$$

then

$$\left| Z_{\pm n}^{(m)}(x) \right| \leqq M^{m+1} \qquad \text{for } \xi(x, \lambda) \text{ in } \Xi_{j,k}. \tag{15}$$

This upper bound exists by virtue of the hypothesis made in §3 and the remarks of the preceding paragraph. From (15) it is clear that the infinite series (14) converges uniformly when $|\lambda| > M$ and ξ is in $\Xi_{j,k}$. Referring to the expressions (12), (10), and (4), equation (14) may now be rewritten in terms of $u_{j,k}(x, \lambda)$ and $y_{\pm n}(x, \lambda)$. In this connection we note that

$$[\lambda/W(x, \lambda)]^{1/2} = B(\lambda) + O(\lambda^{-n-1}),$$

where $B(\lambda)$ is a bounded function of λ.

To summarize the conclusions of our discussion we state the following

THEOREM. *Under the hypotheses stated in* §§1 *and* 3, *over each region* $\Xi_{j,k}$ *on* R_ξ *the given differential equation* (2) *has a solution* $u_{j,k}(x, \lambda)$ *of the form*

$$u_{j,k}(x, \lambda) = e^{\pm\xi}\left[A_{\pm n}(x, \lambda) + \frac{O(\lambda^{-n-1})}{\phi^{1/2}}\right], \qquad j = 1, 2, k = 0, \pm 1, \cdots.$$

The functions $\xi(x, \lambda)$, $A_{\pm n}(x, \lambda)$, and $\phi(x)$ and the region $\Xi_{j,k}$ are described by the several formulas (3), (5), (7), (8), and (9). Since the regions $\Xi_{1,k}$ and $\Xi_{1,k-1}$ cover the region $\Xi_{2,k}$, the theorem determines a pair of linearly independent solutions of (2) for each $x \neq 0$ in R_x.

The asymptotic representation of a solution $u_{j,k}(x, \lambda)$ given by the theorem does not in general hold over all of R_x but only on the image of $\Xi_{j,k}$.

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ON STIELTJES INTEGRATION

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Among the theorems concerning the Stieltjes integral there are two which are established for integrals in one-dimensional space, but not in spaces of more than one dimension. These are (I) if $\int f dg$ exists, f and g have no common discontinuity; (II) if $\int f dg$ exists, and g is of bounded variation and t is its total-variation function, then $\int f dt$ exists. The method of proof for one dimension[1] does not extend to higher dimensions. In this note extensions of these theorems to n dimensions are proved for the ordinary Stieltjes integral and for a modified form of it.[2]

1. **Definitions.** Throughout this note we shall assume that f is real-valued and bounded on a set D in the space R^n, and that g is real-valued on R^n. For each interval $I \subset R^n$ we define $\Delta_g I$ in the usual way, as the sum of 2^n terms each of which is ± 1 times the value of g at a vertex of I. If B is a closed interval contained in D, an *extended partition* of B is a set $P = \{I_1, I_2, \cdots, I_k, x_1, x_2, \cdots, x_k\}$ in

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[1] L. M. Graves, *Theory of functions of real variables*, McGraw-Hill, 1946, p. 263, Theorem 4, and p. 273, Theorem 14.

[2] E. J. McShane and T. A. Botts, *A modified Riemann-Stieltjes integral*, Duke Math. J. vol. 19 (1952) p. 293.