ON SOME PROPERTIES OF THE NORMAL AND GAMMA DISTRIBUTIONS

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It is well-known [2] that in a normal population any translation-invariant statistic is stochastically independent of the sample mean. Similarly [3] in a gamma population the sample mean is distributed independently of any scale-invariant statistic.

In the present note we shall prove the following theorems.

THEOREM 1. *Let $x_1, x_2, \cdots, x_n$ be n identically and independently distributed normal variables. The necessary and sufficient condition that the sum $x_1+x_2+\cdots+x_n$ is distributed independently of some function $g(x_1, x_2, \cdots, x_n)$ is that $g(x_1, x_2, \cdots, x_n)$ and $g(x_1+\lambda, x_2+\lambda, \cdots, x_n+\lambda)$ should be identically distributed.*

PROOF OF NECESSITY. From the conditions of Theorem 1, it follows that

$$E\left(\exp\left(it\sum x + iug\right)\right) = E\left(\exp\left(it\sum x\right)\right)\cdot E(\exp(iug)) \tag{1}$$

where $i=(-1)^{1/2}$ as usual and t and u are real.

Without any loss of generality, we may assume x's to be distributed normally with zero mean and unit variance.

Then (1) gives

$$\begin{aligned} E\{\exp(iug(x_1, x_2, \cdots x_n))\} &= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big(iug(x_1, x_2, \cdots x_n) \\ &\quad - \frac{1}{2}\sum(x-it)^2\Big)dx_1\cdots dx_n = \phi(t, u), \text{ say.}\end{aligned} \tag{2}$$

Now from (2) it follows that $\phi(t, u)$ does not involve t for all real values of t.

But it is evident that $\phi(t, u)$ is an entire function in t and hence it should be free of t for all complex values of t also.

Hence putting $t=-i\lambda$ (λ any real number) we have, after some simplifications,

$$\phi(-i\lambda, u) = E\{\exp(iug(x_1+\lambda, \cdots, x_n+\lambda))\}. \tag{3}$$

Thus combining (2) and (3) we have

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$$E\{\exp(iug(x_1, x_2, \cdots, x_n))\} = E\{\exp(iug(x_1+\lambda, x_2+\lambda, \cdots, x_n+\lambda))\}. \tag{4}$$

Then using the uniqueness theorem of characteristic functions [1], it follows from (4) that $g(x_1, x_2, \cdots, x_n)$ and $g(x_1+\lambda, x_2+\lambda, \cdots, x_n+\lambda)$ should be identically distributed.

PROOF OF SUFFICIENCY. As before let us assume x's to be distributed normally with zero mean and unit variance.

Then the characteristic function of the joint cumulative distribution of $\sum x$ and $g(x_1, x_2, \cdots, x_n)$ is given by

$$\begin{aligned} E\left(\exp\left(it\sum x + iug\right)\right) &= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(it\sum x + iug(x_1, \cdots, x_n) - \frac{1}{2}\sum x^2\right)dx_1\cdots dx_n \\ &= E\left(\exp\left(it\sum x\right)\right)\cdot\phi(t, u), \text{ say,} \end{aligned} \tag{5}$$

where

$$\phi(t, u) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(iug(x_1, \cdots, x_n) - \frac{1}{2}\sum (x - it)^2\right)dx_1\cdots dx_n.$$

Now it is given that $g(x_1, x_2, \cdots, x_n)$ and $g(x_1+\lambda, x_2+\lambda, \cdots, x_n+\lambda)$ are identically distributed and hence their corresponding characteristic functions should be identical, that is,

$$\begin{aligned} E\{\exp(iug(x_1, x_2, \cdots, x_n))\} &= E\{\exp(iug(x_1+\lambda, x_2+\lambda, \cdots, x_n+\lambda))\} \\ &= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(iug(x_1+\lambda, \cdots, x_n+\lambda) - \frac{1}{2}\sum x^2\right)dx_1\cdots dx_n \\ &= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(iug(x_1, \cdots, x_n) - \frac{1}{2}\sum (x-\lambda)^2\right)dx_1\cdots dx_n \\ &= \phi(-i\lambda, u). \end{aligned} \tag{6}$$

Thus it follows from (6) that $\phi(t, u)$ which is defined only for real t and real u in (5) must also exist for all complex t, since then it represents the characteristic function of the distribution of $g(x_1, x_2, \cdots, x_n)$ under the assumptions of the theorem.

Again from (6) it is obvious that $\phi(t, u)$ does not involve t for all complex t. Hence $\phi(t, u)$ being an entire function in t should be free of t for all real t also.

Thus we have the relation

$$(7) \qquad E\left(\exp\left(it \sum x + iug\right)\right) = E\left(\exp\left(it \sum x\right)\right) \cdot E(\exp(iug)).$$

Then the stochastic independence of $x_1+x_2+\cdots+x_n$ and $g(x_1, x_2, \cdots, x_n)$ follows immediately from (7).

Proceeding exactly in the same way we can prove a corresponding theorem for the case of gamma variates which may be stated as follows:

THEOREM 2. *Let $x_1, x_2, \cdots, x_n$ be n independently and identically distributed gamma variates with the distribution function*

$$dF(x) = \frac{1}{\Gamma(p)} e^{-x} x^{p-1} dx \qquad (0 \leqq x \leqq \infty).$$

Then the necessary and sufficient condition that the sum $\sum x$ is distributed independently of some function $g(x_1, \cdots, x_n)$ is that $g(x_1, \cdots, x_n)$ and $g(\lambda x_1, \cdots, \lambda x_n)$ should be identically distributed.

In conclusion, the author expresses his thanks to the referee for some valuable comments.

Note added in proof: It is easy to construct examples to show that for the case of independent observations from a normal (or gamma) distribution, some statistic may be distributed independently of the mean without being translation- (or scale-) invariant. A number of such examples have been given by the author in his doctoral thesis *On characterizations of probability distributions and statistics*, submitted to Calcutta University in 1955.

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