A NOTE ON TORSION-FREE NIL GROUPS

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Introduction. A *nil group* is an (additively written) abelian group G such that every associative ring R with G as its additive group has trivial multiplication, i.e., $xy=0$ for every $x, y \in R$. Szele [2] has shown that every nil group is either a torsion group (every element of finite order) or a torsion-free group (every element except the identity of infinite order), and he has completely characterized the torsion nil groups. In [3], Szele discusses some questions related to nil groups, but is unable to find conditions for the existence of torsion-free nil groups.

An additive group G is said to be *strongly nil* if there is no nontrivial ring, associative or nonassociative, which has G as its additive group. It is the purpose of this note to characterize those nil and strongly nil groups which are (isomorphic to) a weak direct sum of subgroups of the additive group of rationals, and to show that every such nil group is also strongly nil.

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The basic lemma. We use, in modified form, the characterization of the subgroups of the additive rationals R^+ given by Beaumont and Zuckerman [1]: designate by $p_1, p_2, \dots, p_j, \dots$ the primes in their natural order; then for any subgroup K of R^+ which contains the rational integers, let k_j be the greatest non-negative integer n such that p_j^{-n} appears in K , if such an integer exists, and the symbol ∞ otherwise. K is, then, the set of all rationals u/v with u an arbitrary integer and v an arbitrary integer of the form $\prod_j p_j^{n_j}$ with $n_j \leq k_j$. We denote K by the sequence $(k_1, k_2, \dots, k_j, \dots)$; and since every nonzero subgroup of R^+ is isomorphic to one containing the rational integers, this characterization is essentially complete. The initials i.c. will designate "integer containing" with reference to subgroups of R^+ .

If $K = (k_1, k_2, \dots)$, $L = (l_1, l_2, \dots)$, $M = (m_1, m_2, \dots)$ are i.c. subgroups of R^+ , define (K, L, M) to be the set of all rationals t such that $tKLC \subset M$. It is clear that (K, L, M) is a subgroup of M , and we shall

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need conditions making $(K, L, M) = 0$. Assume $(K, L, M) \neq 0$, then

(1) if $k_j = \infty$, then $m_j = \infty$,

(2) if $l_j = \infty$, then $m_j = \infty$,

(3) if k_j and l_j are both finite, then $m_j \geq k_j + l_j$ for almost all j .

For if $k_j = \infty$ and $m_j < \infty$, every element of (K, L, M) would be divisible by p_j^n for arbitrarily high powers n , and this is a contradiction unless $(K, L, M) = 0$. The proof of (2) is analogous. If $k_j < \infty$ and $l_j < \infty$, and if there were infinitely many subscripts j for which $m_j < k_j + l_j$, every element of (K, L, M) would have to be divisible by infinitely many primes, yielding again $(K, L, M) = 0$. This proves (3).

On the other hand, if (1), (2), (3) hold, and if $m_j \geq k_j + l_j$ for all j , then (K, L, M) contains 1. Otherwise, if $j_1, \dots, j_q$ denote those subscripts for which $k_{j_i} < \infty$, $l_{j_i} < \infty$, $m_{j_i} < k_{j_i} + l_{j_i}$, $i = 1, \dots, q$, then

$$\prod_{i=1}^q p_{j_i}^{n_i} \in (K, L, M)$$

where $n_i = k_{j_i} + l_{j_i} - m_{j_i}$, and again $(K, L, M) \neq 0$.

LEMMA. $(K, L, M) = 0$ if, and only if, (1) or (2) or (3) is false.

(As an easy application of this lemma, suppose K is an i.c. nil subgroup of R^+ . Then $(K, K, K) = 0$, and since (1) and (2) are satisfied, (3) must be false, i.e., $k_j < 2k_j$ for an infinite number of subscripts j . Hence, the i.c. nil subgroups of R^+ are given by those sequences $(k_1, k_2, \dots)$ for which $0 < k_j < \infty$ for an infinite number of k_j 's. This is essentially Theorem 3 of [1].)

Direct sum case. Let Λ be any index set, and let $\{H_\lambda\}$, $\lambda \in \Lambda$, be a set of i.c. subgroups of R^+ . The weak direct sum G of these groups H_λ is defined in terms of a basis $\{a_\lambda\}$, $a_\lambda \in H_\lambda$, as the set of all elements

$$x = \sum_{\lambda} r_\lambda(x) a_\lambda$$

where: $r_\lambda(x) \in H_\lambda$; for each x , $r_\lambda(x) = 0$ for almost all $\lambda \in \Lambda$; equality and addition are defined componentwise; $0a_\lambda = 0$ and $1a_\lambda = a_\lambda$ for all $\lambda \in \Lambda$. If R is a ring with G as its additive group, then

$$a_\lambda a_\mu = \sum_{\nu} s'_{\lambda\mu} a_\nu \in R$$

for suitable structure constants $s'_{\lambda\mu} \in H_\nu$, and

$$r_\nu(xy) = \sum_{\lambda, \mu} r_\lambda(x) r_\mu(y) s'_{\lambda\mu}.$$

THEOREM 1. G is strongly nil if and only if $(H_\lambda, H_\mu, H_\nu) = 0$ for every λ, μ, ν in Λ .

THEOREM 2. G is nil if and only if $(H_\lambda, H_\mu, H_\nu) = 0$ for every λ, μ, ν in Λ satisfying either (i) $\lambda = \mu = \nu$ or (ii) $\lambda \neq \nu, \mu \neq \nu$.

PROOFS. Assume $t \neq 0$ is an element of $(H_\lambda, H_\mu, H_\nu)$ for some λ, μ, ν of Λ . For each pair of elements $x, y \in G$, define multiplication by

$$xy = tr_\lambda(x)r_\mu(y)a_\nu$$

and this makes G into a nontrivial ring R (since $a_\lambda a_\mu \neq 0$). Associativity of this multiplication is implied by (i) or (ii):

$$(xy)z = tr_\lambda(tr_\lambda(x)r_\mu(y)a_\nu)r_\mu(z)a_\nu,$$

$$x(yz) = tr_\lambda(x)r_\mu(tr_\lambda(y)r_\mu(z)a_\nu)a_\nu.$$

If $\lambda = \mu = \nu$, each of these becomes $t^2 r_\lambda(x)r_\lambda(y)r_\lambda(z)a_\lambda$. If $\lambda \neq \nu, \mu \neq \nu$, $r_\lambda(tr_\lambda(x)r_\mu(y)a_\nu) = r_\mu(tr_\lambda(y)r_\mu(z)a_\nu) = 0$.

Conversely, let R be a ring over G , and assume $(H_\lambda, H_\mu, H_\nu) = 0$ for some λ, μ, ν . It is clear that $s_{\lambda\mu}^\nu = 0$, then, from which it follows that G is strongly nil. However, if $s_{\lambda\mu}^\nu = 0$ under conditions (i) or (ii), then $a_\lambda a_\lambda = 0$ for every λ , and $a_\lambda a_\mu = s_{\lambda\mu}^\lambda a_\lambda + s_{\lambda\mu}^\mu a_\mu$. The associative law imposed on the product $(a_\lambda a_\mu)a_\mu$ yields $s_{\lambda\mu}^\mu = 0$, and a similar argument on the product $a_\lambda(a_\lambda a_\mu)$ yields $s_{\lambda\mu}^\lambda = 0$. Therefore, G is nil.

Remark. It is interesting that for groups which are the weak direct sum of i.c. subgroups of R^+ , "nil" implies "strongly nil." For suppose t is a nonzero element of (K, L, K) . Then $tKL \subset K$, and so $tKL^2 \subset KL$, whence $t^2KL^2 \subset tKL \subset K$. It follows that t^2 is an element of (L, L, K) . Similarly, $(K, L, L) \neq 0$ implies $(K, K, L) \neq 0$. Thus, the conditions of Theorem 2 imply those of Theorem 1.

REFERENCES

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