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A NOTE ON RECURRENT FLOWS

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Let R be a metric space and let $f: R \times \{t \mid t \text{ is real}\} \rightarrow R$ be a continuous flow in R . The flow f is said to be *recurrent* (in the sense of Gottschalk and Hedlund) provided that if ϵ and s are positive real numbers, then there exists a real number t such that $t > s$ and $\rho(x, f(x, t)) < \epsilon$ for all $x \in R$.

THEOREM. Consider the statements:

- (I) f is recurrent,
- (II) There exists a sequence $t_1 < t_2 < \dots \rightarrow +\infty$ of real numbers such that $\lim_{n \rightarrow \infty} f(A, t_n) = A$ for every closed subset A of R ,
- (III) There exists a sequence $t_1 < t_2 < \dots \rightarrow +\infty$ of real numbers such that $\limsup_{n \rightarrow \infty} f(A, t_n) \subset A$ for every closed subset A of R .

Then,

- (1) I implies II; II implies III,
- (2) If R is compact, then I, II, III are equivalent.

PROOF. Assume I. There exists a sequence $t_1 < t_2 < \dots \rightarrow +\infty$ of real numbers such that $\rho(x, f(x, t_n)) < 1/n$ for all $x \in R$ and all positive integers n . Let A be a closed subset of R . If $x \in A$, then $f(x, t_n) \rightarrow x$. Hence $A \subset \liminf_{n \rightarrow \infty} f(A, t_n)$. Suppose $x \in \limsup_{n \rightarrow \infty} f(A, t_n)$. There exist $x_1, x_2, \dots \in A$ and a sequence $n_1 < n_2 < \dots \rightarrow +\infty$ of positive integers such that $\rho(x, f(x_i, t_{n_i})) \rightarrow 0$. Since $\rho(x_i, f(x_i, t_{n_i})) < 1/n_i$ for all positive integers i , it follows that $\rho(x_i, f(x_i, t_{n_i})) \rightarrow 0$ whence $x_i \rightarrow x$ and $x \in A$. Hence $\limsup_{n \rightarrow \infty} f(A, t_n) \subset A$. This proves that I implies II.

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It is immediate that II implies III.

Assume III. Let R be compact. Let ϵ and s be positive real numbers. There exists a finite sequence $A_1, \dots, A_k$ of closed subsets of R such that $A_1 \cup \dots \cup A_k = R$ and $\text{diam } (A_i) < \epsilon/2$ ($i=1, \dots, k$). Choose a positive integer n so that $t_n > s$ and $f(A_i, t_n) \subset N(A_i, \epsilon/2)$ ($i=1, \dots, k$) where $N(A, \delta)$ denotes the δ -neighborhood of the set A . Let $x \in R$. Then $x \in A_j$ for some j ($j=1, \dots, k$). It follows that $\rho(f(x, t_n), y) < \epsilon/2$ for some $y \in A_j$. Since $x, y \in A_j$ we have $\rho(x, y) < \epsilon/2$. Hence, $\rho(x, f(x, t_n)) < \epsilon$. This proves (2).

We remark that the above theorem and proof apply to discrete flows. Also the theorem answers a question by Nemyckii concerning a characterization of pseudo-minimal sets (cf. V. V. Nemyckii, *Topological problems of the theory of dynamical systems*, American Mathematical Society Translation, no. 103, p. 69).

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