A NOTE ON TWO GENERATING FUNCTIONS FOR LEGENDRE FUNCTIONS

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It has recently been shown by Bloch [1] that for Legendre functions not on the cut

$$(1) \quad e^{tz} I_m [t(z^2 - 1)^{1/2}] = \sum_{n=0}^{\infty} P_{m+n}^m(z) \frac{t^{m+n}}{(2m+n)!}, \quad m = 0, 1, 2, \dots,$$

where

$$I_m(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{m+2n}}{n!(m+n)!}, \quad P_n^m(z) = (z^2 - 1)^{m/2} d^m P_n(z)/dz^m.$$

It is the purpose of this note to show how (1) and the result (2) below may be derived using methods of Truesdell [2] and thus to correct the results (58) and (66) on pages 101 and 105 of [2].¹

If one applies Theorem 14.4 of [2] to the function

$$F(z, \alpha) = \frac{\cos \pi(\alpha - m)}{\Gamma(1 - \alpha + 2m)} (z^2 - 1)^{-(\alpha-m)/2} P_{m-\alpha}^m \left[\frac{-z}{(z^2 - 1)^{1/2}} \right],$$

the changes of variable $x = -z(z^2 - 1)^{-1/2}$, $yz = xt$ lead after simplifications to the generating function relationship (1). Application of Theorem 14.7 of [2] to the function

$$F(z, \alpha) = \Gamma(\alpha - b + 1)(z^2 - 1)^{-(\alpha+1)/2} P_{\alpha}^b \left[\frac{-z}{(z^2 - 1)^{1/2}} \right]$$

gives the formula

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n - b + 1)}{n! \Gamma(\alpha + n + 1)} (z^2 - 1)^{-(\alpha+n+1)/2} P_{\alpha+n}^b \left[\frac{-z}{(z^2 - 1)^{1/2}} \right] y^n \\ = \frac{\Gamma(\alpha - b + 1)}{2\pi i} \int_c e^w w^{-\alpha} [(z + y/w)^2 - 1]^{-(\alpha+1)/2} \\ \cdot P_{\alpha}^b \left[\frac{-z - y/w}{\{(z + y/w)^2 - 1\}^{1/2}} \right] dw, \end{aligned}$$

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¹ Truesdell's results are incorrect because of computational error. The incorrect formula 19.10(17) of [3] should also be replaced by (1).

where the contour of integration extends from $-\infty$ counterclockwise about the points $y/(1-z)$ and $-y/(1+z)$ and back to $-\infty$. Upon making the substitutions $\alpha = b = m$, $m = 0, 1, 2, \dots$, $x = -z(z^2 - 1)^{-1/2}$, $y(z^2 - 1)^{-1/2} = t$, and $w = ts$, one finds after simplifications that for the Legendre functions away from the cut

$$(2) \quad \frac{(2m-1)!!(x^2-1)^{m/2}}{2\pi i} \int_C e^{st}(1-2xs+s^2)^{-m-1/2} s^m ds \\ = \sum_{n=0}^{\infty} P_{n+m}^m(x) \frac{t^{m+n}}{(m+n)!}.$$

The variables y and t are real. The contour C now extends from $-\infty$ counterclockwise around the zeros of $(1-2xs+s^2)$ and back to $-\infty$.

REFERENCES

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2. C. Truesdell, *A unified theory of special functions*, Princeton University Press, 1948.
3. The Bateman Manuscript Project, *Higher transcendental functions*, vol. 3, McGraw-Hill, 1955.

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