# THE EXISTENCE OF A VECTOR OF WEIGHT 0 IN IRREDUCIBLE LIE GROUPS WITHOUT CENTRE

HANS FREUDENTHAL

R. Bott and H. Samelson raised the question (communicated to me by W. T. van Est) whether it is true that every irreducible linear, semi-simple compact Lie group with trivial centre possesses a vector of weight 0. In the following I shall give a proof of this conjecture and of its converse. In the meantime R. Bott wrote me that Harish-Chandra also proved this conjecture. His proof is quite different from the present one.

Let $G$ be an irreducible linear, semi-simple compact Lie group.

THEOREM. *The centre of $G$ is trivial if and only if $G$ possesses a vector of weight* 0.

Let $H$ be a Cartan subgroup of $G$. Since $G$ is semi-simple and compact every element of $G$ is conjugate to an element of $H$.[1] Therefore the centre $C$ of $G$ is contained in $H$.

Furthermore every element of $H$ is generated by an infinitesimal element of $H$.

The centre $C$ of $G$ consists of those elements of $G$ which in the adjoint representation ad $G$ are represented by 1.

Therefore if $h$ is an infinitesimal element of $H$, then exp $h \in C$ if and only if

$$\alpha(\text{ad } h) \equiv 0 \mod 2\pi i$$

for each root form $\alpha$.

This central element is trivial, if and only if

$$\omega(\text{ad } h) \equiv 0 \mod 2\pi i$$

for each weight $\omega$ of $G$.

Therefore there is equivalence between the two assertions:

(1) $C$ is trivial.

(2) $\alpha(\text{ad } h) \equiv 0 \mod 2\pi i$ for all root forms implies $\omega(\text{ad } h) \equiv 0 \mod 2\pi i$ for all weights.

The second assertion may also be stated in the form:

(3) Every weight is a linear combination of root forms with integer coefficients.

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[1] E. Cartan, Annali di Matematica Pura ed Applicata (4) vol. 4 p. 214.

We must now show that (3) is equivalent to:

(4) There is a weight 0.

Proof (3)→(4): We choose the weight $\omega$ and the representation

$$\omega = \sum_{\nu=1}^{p} \alpha_\nu$$

by a system of root forms $\alpha_\nu$ so that $p$ is minimal. If $p>0$, then under the Weyl normalization ($N_\alpha = -1$) of the Killing-Cartan quadratic form written as an inner product, we have

$$(\alpha, \alpha) < 0 \quad \text{and} \quad (\omega, \omega) < 0$$

for all root forms and weights.

$$(\omega, \omega) = \sum_\nu (\omega, \alpha_\nu).$$

Therefore

$$(\omega, \alpha_\nu) < 0$$

for at least one $\nu$, and

$$2\frac{(\omega, \alpha_\nu)}{(\alpha_\nu, \alpha_\nu)} \geqq 1.$$

This implies that $\omega - \alpha_\nu$ is also a weight, in contradiction to the minimality property of $p$. Therefore $p$ must be 0, and 0 must be a weight.

(4)→(3): Because of the irreducibility of $G$ every weight can be derived from the highest weight by successive subtraction of root forms. Therefore if 0 is a weight, then the highest weight is a sum of root forms and then every other weight has the same property.

Mathematical Institute, Utrecht