ON THE ABSOLUTE LOGARITHMIC SUMMABILITY OF A FOURIER SERIES AND ITS DIFFERENTIATED SERIES

R. MOHANTY AND S. MAHAPATRA

1. Let $f(t)$ be integrable L and periodic with period 2π , and let

$$(1.1) \quad f(t) \sim \frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_1^{\infty} A_n(t).$$

Then the series obtained by formally differentiating (1.1) at $t=x$ is

$$(1.2) \quad \sum_1^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum nB_n(x).$$

We write

$$(1.3) \quad \phi(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\},$$

$$(1.4) \quad \psi(t) = \frac{1}{2} \{f(x+t) - f(x-t)\},$$

$$(1.5) \quad g(t) = \psi(t)/t,$$

$$(1.6) \quad \Phi(t) = \int_t^{\pi} \phi(u)/u du,$$

$$(1.7) \quad G(t) = \int_t^{\pi} g(u)/u du,$$

$$(1.8) \quad \phi_1(t) = \frac{1}{t} \int_0^t \phi(u) du.$$

DEFINITION. Let $\lambda(w)$ be continuous, differentiable, and monotone increasing in (A, ∞) , where A is some positive number, and let $\lambda(w) \rightarrow \infty$ as $w \rightarrow \infty$.

A series $\sum a_n$ is summable $|R, \lambda, r|$ where $r > 0$, if

$$\int_A^{\infty} \left| \frac{r\lambda'(w)}{\{\lambda(w)\}^{r+1}} \sum_{n < w} \{\lambda(w) - \lambda(n)\}^{r-1} a_n \lambda_n \right| dw < \infty \quad [3].$$

The object of the present paper is to prove the following theorems.

Received by the editors April 21, 1955.

THEOREM A. If¹

$$\Phi(t)/\log(k/t)$$

is of bounded variation in $(0, \pi)$, then (1.1) is summable $|R, \log w, 2|$ at the point $t=x$.

THEOREM B. If

$$G(t)/\log(k/t)$$

is of bounded variation in $(0, \pi)$, then (1.2) is summable $|R, \log w, 2|$.

NOTE. The order of summability in the conclusion of Theorem A can not be replaced by 1, as $|R, \log \omega, 1|$ summability of a Fourier series is a nonlocal property of its generating function [4] and therefore can not be ensured by the hypothesis of Theorem A, which is obviously a local one. It can probably be replaced by $1+\delta$, $\delta>0$, but we have not been able to do this. The relation between the condition of Theorem A and a known condition like

$$(1.9) \quad \phi_1(t) \text{ is of bounded variation in } (0, \pi)$$

is contained in

LEMMA 1. If (1.9) holds, then the condition of Theorem A also holds.

We have, by integration by parts,

$$\frac{1}{\log(k/t)} \int_t^\pi \frac{\phi(u)}{u} du = \frac{\phi_1(\pi) - \phi_1(t)}{\log(k/t)} + \frac{1}{\log(k/t)} \int_t^\pi \frac{\phi_1(u)}{u} du.$$

The first term on the right is easily dispensed with. The lemma will be proved if

$$(\log(k/t))^{-1} \int_t^\pi \phi_1(u)/u du$$

is of bounded variation in $(0, \pi)$, whenever $\phi_1(t)$ is of bounded variation in $(0, \pi)$, and, since the former is the mean value of the latter, the result follows by using familiar arguments [5].

2. In the proof of Theorem A, we require the following inequalities for the kernel $h(w, t) = \sum_{n < w} \log n \log(w/n) \cos nt$:

$$(2.1) \quad h(w, t) = O\{(\log w)t^{-1}\},$$

$$(2.2) \quad h(w, t) = O\{w(\log w)^2\}.$$

PROOF OF (2.1). Let $m \leq w < m+1$. We have, by partial summation,

¹ k is a constant greater than π , introduced merely to avoid the singularity of $(\log 1/t)^{-1}$ at $t=1$.

$$\begin{aligned}
& \sum_1^m \log n \log \frac{w}{n} \cos nt \\
&= \sum_1^{m-1} \log \left(1 + \frac{1}{n}\right) \left\{ \log n + \log(n+1) - \log w \right\} \frac{\sin(n+1/2)t}{2 \sin(t/2)} \\
&\quad + \log m \log \frac{w}{m} \frac{\sin(m+1/2)t}{2 \sin(t/2)} \\
&= O \left\{ t^{-1} \log w \max_{N, N'} \left| \sum_N^{N'} \log \left(1 + \frac{1}{n}\right) \sin \left(n + \frac{1}{2}\right)t \right| \right\} \\
&\quad + O \left\{ \frac{\log w}{w} t^{-1} \right\} \\
&= O \{ t^{-1} \log w \},
\end{aligned}$$

on using $\log(1 + 1/n) = 1/n + O(1/n^2)$.

Proof of (2.2) is trivial.

3. PROOF OF THEOREM A. We have

$$\begin{aligned}
(3.1) \quad A_n(x) &= \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \, dt = \frac{2}{\pi} \int_0^\pi t \Phi'(t) \cos nt \, dt \\
&= -\frac{2}{\pi} \int_0^\pi d \left\{ \frac{\Phi(t)}{\log(k/t)} \right\} \left\{ t \log \frac{k}{t} \cos nt + \frac{\sin nt}{n} \right\}
\end{aligned}$$

(on integrating by parts twice in succession)

$$\begin{aligned}
&= \frac{2}{\pi} \int_0^\pi d \left\{ \frac{\Phi(t)}{\log(k/t)} \right\} t \log \frac{k}{t} \cos nt \, dt + \frac{2}{\pi} \int_0^\pi d \left\{ \frac{\Phi(t)}{(\log k/t)} \right\} \frac{\sin nt}{n} \, dt \\
&= u_n + v_n, \text{ say.}
\end{aligned}$$

The series $\sum u_n$ is summable $|R, \log w, 2|$, if

$$I = \frac{2}{\pi} \int_0^\pi \frac{2dw}{w(\log w)^3} \left| \sum_{n < w} \log n \log \frac{w}{n} u_n \right| < \infty.$$

Substituting for u_n from (3.1) it is easy to see that

$$(3.2) \quad I \leq \frac{4}{\pi} \int_0^\pi J \left| d \left\{ \frac{\Phi(t)}{\log(k/t)} \right\} \right| t \log \frac{k}{t},$$

where

$$J = \int_A^\infty w^{-1} (\log w)^{-3} |h(w, t)| \, dw.$$

Since $\int_0^\pi |d\{\Phi(t)/\log(k/t)\}|$ is finite it is enough to show that

$$(3.3) \quad J = O\left\{\left(\log \frac{k}{t}\right)^{-1} t^{-1}\right\} \quad \text{for } 0 < t < \pi.$$

Writing

$$(3.4) \quad J = \int_A^{k/t} + \int_{k/t}^\infty = J_1 + J_2, \text{ say,}$$

we have, by using (2.2),²

$$(3.5) \quad J_1 < C \int_A^{k/t} \frac{dw}{\log w} = O\left\{t^{-1} \left(\log \frac{k}{t}\right)^{-1}\right\},$$

and by using (2.1)

$$(3.6) \quad J_2 < C \int_{k/t}^\infty \frac{dw}{w(\log w)^2} = O\left\{t^{-1} \left(\log \frac{k}{t}\right)^{-1}\right\},$$

and therefore, by (3.4), (3.5), and (3.6),

$$(3.7) \quad J = O\left\{t^{-1} \left(\log \frac{k}{t}\right)^{-1}\right\},$$

and hence, by (3.2) and (3.7),

$$I \leq C \int_0^\pi \left| d\left\{ \frac{\Phi(t)}{\log(k/t)} \right\} \right| = O(1).$$

Now

$$v_n = \frac{2}{\pi} \int_0^\pi d\left\{ \frac{\Phi(t)}{\log(k/t)} \right\} \frac{\sin nt}{n} dt.$$

Since $\{\Phi(t)/\log(k/t)\}$ is of bounded variation in $(0, \pi)$, $\sum v_n$ is summable $|C, \delta|$, $\delta > 0$ [1]. Hence it is summable $|C, 2|$ and a fortiori summable $|R, \log w, 2|$.

Thus the proof of Theorem A is complete.

4. An interesting corollary of Theorem A is

THEOREM A₁. *The Fourier series of $f(t)$ at $t=x$ is summable $|R, \log w, 2|$ if*

$$(4.1) \quad \int_0^\delta \frac{|\phi(t)|}{t \log(k/t)} dt < \infty.$$

² C is a constant not necessarily the same at each occurrence.

This follows from Theorem A by using

LEMMA 2. *If (4.1) holds, then the condition of Theorem A also holds.*

PROOF OF LEMMA 2. If

$$\theta(t) = \int_t^\pi \phi(u)/u \log(k/u) du$$

then (4.1) is equivalent to

$$(4.2) \quad \theta(t) \text{ is of bounded variation in } (0, \delta),$$

$$\begin{aligned} \left(\log \frac{k}{t}\right)^{-1} \int_t^\pi \frac{\phi(u)}{u} du &= - \left(\log \frac{k}{t}\right)^{-1} \int_t^\pi \log \frac{k}{u} \theta(u) du \\ &= \theta(t) - \left(\log \frac{k}{t}\right)^{-1} \int_t^\pi \frac{\theta(u)}{u} du. \end{aligned}$$

The second term on the right is the mean value of the first, and the conclusion of Lemma 2 follows at once.

NOTE. It follows from Theorem A₁ that if $\phi(t) = O\{(\log k/t)^{-1}\}$, then $\sum A_n(x)$ is summable $|R, \log w, 2|$, but this condition alone is not sufficient for the convergence of $\sum A_n(x)$ [6].

5. PROOF OF THEOREM B. Integrating by parts twice in succession, we have

$$\begin{aligned} nB_n(x) &= -\frac{2}{\pi} \int_0^\pi G'(t) nt^2 \sin ntdt \\ &= -\frac{2}{\pi} \int_0^\pi d \left\{ \frac{G(t)}{\log(k/t)} \right\} \left\{ t^2 \log \frac{k}{t} n \sin nt + t \cos nt - \frac{\sin nt}{n} \right\} \\ &= -\frac{2}{\pi} \int_0^\pi d \left\{ \frac{G(t)}{\log(k/t)} \right\} t^2 \log \frac{k}{t} n \sin nt dt \\ (5.1) \quad &\quad - \frac{2}{\pi} \int_0^\pi d \left\{ \frac{G(t)}{\log(k/t)} \right\} t \cos nt dt \\ &\quad - \frac{2}{\pi} \int_0^\pi d \left\{ \frac{G(t)}{\log(k/t)} \right\} \frac{\sin nt}{n} dt \\ &= \alpha_n + \beta_n + \gamma_n, \text{ say.} \end{aligned}$$

$\sum \alpha_n$ is summable $|R, \log w, 2|$, if

$$I_1 = \int_A^\infty \frac{2dw}{w(\log w)^3} \left| \sum_{n < w} \log n \log \frac{w}{n} \alpha_n \right| < \infty.$$

Now

$$(5.2) \quad I_1 \leq \frac{4}{\pi} \int_0^{\pi} \left| d \left\{ \frac{G(t)}{\log(k/t)} \right\} \right| t^2 \log \frac{k}{t} \int_A^{\infty} \frac{dw}{w(\log w)^3} |F(w, t)|$$

where $F(w, t) = \sum_{n < w} \log n \log(w/n) n \sin nt$.

Then we have

$$(5.3) \quad F(w, t) = O\{(\log w)t^{-2}\},$$

$$(5.4) \quad F(w, t) = O(w^2(\log w)^2).$$

The proof of (5.3) is similar to that of (2.1), whereas the proof of (5.4) is trivial.

Now

$$\int_A^{\infty} w^{-1}(\log w)^{-3} |F(w, t)| dw = \int_A^{k/t} + \int_{k/t}^{\infty} = I_2 + I_3, \text{ say.}$$

Employing (5.4),

$$(5.5) \quad I_2 < C \int_A^{k/t} \frac{w dw}{\log w} = O\left\{t^{-2} \left(\log \frac{k}{t}\right)^{-1}\right\}.$$

Employing (5.3),

$$(5.6) \quad I_3 < Ct^{-2} \int_{k/t}^{\infty} \frac{dw}{w(\log w)^2} = O\left\{t^{-2} \left(\log \frac{k}{t}\right)^{-1}\right\}.$$

By using (5.5) and (5.6) we have $I_1 = O(1)$.

Employing arguments similar to those used in the proof of a known result [2], it can be proved that $\sum \beta_n$ is summable $|C, \beta|$, $\beta > 1$, and hence summable $|C, 2|$ and a fortiori summable $|R, \log w, 2|$. The discussion of $\sum \gamma_n$ is similar to that of $\sum v_n$ in the proof of Theorem A with $G(t)$ in place of $\Phi(t)$ and thus the proof of Theorem B is complete.

REFERENCES

1. L. S. Bosanquet, *Note on the absolute summability (C) of a Fourier series*, J. London Math. Soc. vol. 11 (1936) pp. 11-16.
2. ———, *The absolute Cesàro summability of a Fourier series, Proof of Theorem 1 with $\beta > 1$* , Proc. London Math. Soc. vol. 41 (1936) pp. 517-528.
3. R. Mohanty, *On the absolute Riesz summability of Fourier series and allied series*, Proc. London Math. Soc. Ser. 2 vol. 52 (1951) p. 297.
4. ———, *On the summability $|R, \log, 1|$ of a Fourier series*, J. London Math. Soc. vol. 25 (1950) pp. 57-72.
5. E. C. Titchmarsh, *Theory of functions*, Oxford, 1939, p. 409.
6. A. Zygmund, *Trigonometrical series*, 1935, p. 35.

RAVENSHAW COLLEGE, CUTTACK, INDIA