A FAMILY OF BEST TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS¹

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Two regions E_1 and E_2 in the complex plane are called *twin convergence regions for the continued fraction*

$$(1) \quad 1 + \frac{c_1^2}{1 + \frac{c_2^2}{1 + \dots}},$$

if the fact that $c_{2n-1} \in E_1$, $c_{2n} \in E_2$, for all $n \geq 1$, is sufficient to insure the convergence of (1). Twin convergence regions E_1 and E_2 are called *best* twin convergence regions, if there do not exist regions $E'_1 \supset E_1$ and $E'_2 \supset E_2$, where at least one of the equalities $E'_1 = E_1$, $E'_2 = E_2$ fails to hold, which are twin convergence regions for the continued fraction (1).

Some time ago one of the authors [1] proved that the continued fraction (1) converges if

$$(2) \quad |c_{2n-1}| \leq \rho, \quad |c_{2n} \pm i| \geq \rho + \epsilon, \quad n \geq 1,$$

where $0 < \rho < 1$ and ϵ is an arbitrary positive quantity. It was also shown in [1] that if the ϵ could be removed in (2) a set of best twin convergence regions would be obtained. It is the purpose of this paper to show that the ϵ can indeed be removed.

Denote by $t_n(z)$ the linear fractional transformation

$$1 + \frac{c_n^2}{z}.$$

It was shown in [1] that if

$$(3) \quad |c_{2n-1}| \leq \rho, \quad |c_{2n} \pm i| \geq \rho \quad \text{for all } n \geq 1,$$

then

$$|1 - t_{2n-1}(z)| \leq \rho \quad \text{for all } |z| \geq \rho,$$

and

$$|t_{2n}(z)| \geq \rho \quad \text{for all } |z - 1| \leq \rho.$$

Set

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$$T_n(z) = t_1(t_2(\cdots t_n(z) \cdots)),$$

and denote by Z_1 and Z_2 the sets defined by $|z| \geq \rho$, $|z-1| \leq \rho$, respectively. The sets

$$K_{2n} = T_{2n}(Z_2), \quad K_{2n-1} = T_{2n-1}(Z_1)$$

are circular regions and $|K_n - 1| \leq \rho$ for all $n \geq 1$. Also $K_{n-1} \supset K_n$ for all $n \geq 2$. Thus we have the well known (see for example [2, pp. 70-79]) "nest of circles." Finally, since $1 \in Z_1$, $1 \in Z_2$ and since the n th approximant of the continued fraction (1) is

$$\frac{A_n}{B_n} = T_n(1),$$

we have for all $n \geq m$, $A_n/B_n \in K_m$.

Let R_n be the radius of K_n . Then $\{R_n\}$ forms a nonincreasing sequence. Hence if it can be shown that a subsequence of $\{R_n\}$ converges to zero it will follow that the continued fraction converges. Our proof consists in showing that $\lim R_{2n} = 0$.

We next observe that a consequence of the fundamental recurrence relation

$$(4) \quad B_n = B_{n-1} + c_n^2 B_{n-2},$$

is that

$$\frac{B_n}{B_{n-1}} = 1 + \frac{c_n^2}{1} + \frac{c_{n-1}^2}{1} + \cdots + \frac{c_2^2}{1}.$$

Thus

$$\frac{B_n}{B_{n-1}} = t_n(\cdots t_2(1) \cdots).$$

From this it follows that

$$(5) \quad \left| \frac{B_{2n-1}}{B_{2n-2}} - 1 \right| \leq \rho, \quad \left| \frac{B_{2n}}{B_{2n-1}} \right| \geq \rho.$$

The inequality (5) allows us to conclude that, if condition (3) holds, no $B_n = 0$. Now set

$$m_n = \min \left| \frac{B_{2n}}{B_{2n-1}} \right|, \quad \text{and} \quad k_n = \max \left| \frac{B_{2n-1}}{B_{2n-2}} - 1 \right|.$$

Condition (5) insures that, for all n , $m_n \geq \rho$ and $k_n \leq \rho$. It is, however,

desirable to obtain sharper estimates, depending on n , for these quantities. This is done in the following:

LEMMA 1. *If the elements c_n satisfy condition (3) then*

$$(6) \quad m_n = \rho(1 + (1 - \rho^2)/(n - \rho(n - 2))).$$

Since

$$\left| \frac{B_{2n-1}}{B_{2n-2}} - 1 \right| = \frac{|c_{2n-1}^2|}{|B_{2n-2}/B_{2n-3}|},$$

we have

$$(7) \quad k_n = \rho^2/m_{n-1}.$$

Equation (6) is readily verified for $n=1$ since

$$m_1 = \min |1 + c_2^2| = \rho(2 - \rho).$$

Now

$$m_n = \min \left| 1 + \frac{c_{2n}^2}{B_{2n-1}/B_{2n-2}} \right|,$$

and one easily sees that the minimum is attained when c_{2n} is of the form $i + \rho e^{i\theta}$ (we can assume that all c_n are in the upper half-plane) and B_{2n-1}/B_{2n-2} is of the form $1 + k_n e^{i\phi}$. Thus

$$m_n = \min \left| 1 + \frac{(i + \rho e^{i\theta})^2}{1 + k_n e^{i\phi}} \right|, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq 2\pi.$$

Now

$$\left| 1 + \frac{(i + \rho e^{i\theta})^2}{1 + k_n e^{i\phi}} \right| = \left| 1 - \frac{(1 - \rho i e^{i\theta})^2}{1 - k_n^2} (1 + k_n e^{i\phi'}) \right|,$$

where ϕ' depends on k_n , ρ , and ϕ , but ranges from 0 to 2π . Hence

$$m_n \geq \frac{1}{1 - k_n^2} (|(1 - k_n^2) - (1 - \rho i e^{i\theta})^2| - k_n |1 - \rho i e^{i\theta}|^2).$$

An elementary argument shows that the minimum of the expression on the right is attained for $\theta = \pi/2$. Thus

$$m_n = \frac{\rho^2 + 2\rho - k_n}{1 + k_n}.$$

With the help of (7) we obtain the recurrence relation

$$m_n = \frac{\rho(2 + \rho)m_{n-1} - \rho^2}{m_{n-1} + \rho^2}.$$

The proof of the lemma is now easily completed by induction.

LEMMA 2. *If the elements c_n satisfy condition (3) then*

$$1 - \rho^2 \left| \frac{B_{2n-1}}{B_{2n}} \right|^2 > \frac{1 + \rho}{n + 1 + r},$$

where

$$r = \rho + 2\rho/(1 - \rho).$$

It follows from Lemma 1 that

$$\begin{aligned} 1 - \rho^2 \left| \frac{B_{2n-1}}{B_{2n}} \right|^2 &\geq 1 - \frac{\rho^2}{\rho^2(1 + (1 - \rho^2)/(n - \rho(n - 2)))^2} \\ &= 1 - \left(\frac{n - \rho(n - 2)}{n - \rho(n - 2) + 1 - \rho^2} \right)^2 \\ &= \left(1 + \frac{n - \rho(n - 2)}{n - \rho(n - 2) + 1 - \rho^2} \right) \left(\frac{1 - \rho^2}{n(1 - \rho) + 2\rho + 1 - \rho^2} \right) \\ &> \frac{1 - \rho^2}{n(1 - \rho) + 2\rho + 1 - \rho^2} = \frac{1 + \rho}{n + 1 + r}. \end{aligned}$$

LEMMA 3. *If the elements c_n satisfy condition (3) and if $c_{2n} \neq 0$, then*

$$\left| \frac{B_{2n}}{c_{2n}B_{2n-2}} \right| = \left| \frac{1}{c_{2n}} + c_{2n} + \theta_n \frac{c_{2n-1}^2}{c_{2n}} \right| \geq \rho(1 + 1/(n + r)),$$

$n = 1, 2, \dots$, provided $B_{-1} = 0$, $B_0 = 1$. Here $r = \rho + 2\rho/(1 - \rho)$, and $\theta_n = B_{2n-3}/B_{2n-2}$.

Applying the recurrence relation (4) twice to B_{2n} we obtain

$$\frac{B_{2n}}{B_{2n-2}} = 1 + c_{2n}^2 + \frac{c_{2n-1}^2}{c_{2n-1}} \cdot B_{2n-3}/B_{2n-2}.$$

Thus if we introduce θ_n and divide by c_{2n} we have

$$\frac{B_{2n}}{c_{2n}B_{2n-2}} = \frac{1}{c_{2n}} + c_{2n} + \theta_n \frac{c_{2n-1}^2}{c_{2n}^2}.$$

Note that, in view of relation (5), $|\theta_n| \leq 1/\rho$.

We now observe that the minimum of the above expression is ob-

tained when the quantities c_n and θ_n lie on the boundaries of the regions over which they are permitted to vary. Hence

$$\begin{aligned} \left| 1/c_{2n} + c_{2n} + \theta_n c_{2n-1}^2/c_{2n} \right| &\geq \left| 1/c_{2n} + c_{2n} \right| - \left| \theta_n c_{2n-1}^2/c_{2n} \right| \\ &\geq \rho \left| \frac{2i + \rho e^{i\theta}}{i + \rho e^{i\theta}} \right| - \frac{\rho^2}{m_n |i + \rho e^{i\theta}|}. \end{aligned}$$

The minimum value is obtained for $\theta = \pi/2$. This leads to

$$\left| 1/c_{2n} + c_{2n} + \theta_n c_{2n-1}^2/c_{2n} \right| \geq \rho \frac{(2 + \rho)}{(1 + \rho)} - \frac{\rho^2}{m_{n-1}(1 + \rho)}$$

which, using the result of Lemma 1, becomes

$$\rho(1 + 1/(n + \rho + 2\rho/(1 - \rho))).$$

We are now ready to prove

THEOREM 1. *The continued fraction (1) converges if for all $n \geq 1$*

$$|c_{2n-1}| \leq \rho, \quad |c_{2n} \pm i| \geq \rho,$$

where $0 < \rho < 1$.

To prove this theorem it suffices, as indicated in the beginning of this paper, to show that $\lim R_{2n} = 0$. It is easily verified that

$$T_{2n}(z) = \frac{A_{2n-1}Z + c_{2n}^2 A_{2n-2}}{B_{2n-1}Z + c_{2n}^2 B_{2n-2}}.$$

From the theory of linear fractional transformations it follows that C_{2n} , the center of K_{2n} , is $T_{2n}(z^*)$, where z^* is the conjugate point with respect to the circle $|z-1| = \rho$ of the point z' , which is defined by $T_{2n}(z') = \infty$. Now

$$z' = -\frac{c_{2n}^2 B_{2n-2}}{B_{2n-1}}.$$

Therefore

$$z^* = \frac{\overline{B}_{2n} - \rho^2 \overline{B}_{2n-1}}{\overline{B}_{2n}},$$

and hence

$$C_{2n} = \frac{A_{2n} \overline{B}_{2n} - \rho^2 A_{2n-1} \overline{B}_{2n-1}}{|B_{2n}|^2 - \rho^2 |B_{2n-1}|^2}.$$

Since $T_{2n}(1-\rho)$ is a point on the boundary of K_{2n} we can write

$$\begin{aligned}
R_{2n} &= |C_{2n} - T_{2n}(1 - \rho)| \\
&= \left| \frac{A_{2n}\overline{B}_{2n} - \rho^2 A_{2n-1}\overline{B}_{2n-1}}{|B_{2n}|^2 - \rho^2 |B_{2n-1}|^2} - \frac{A_{2n} - \rho A_{2n-1}}{B_{2n} - \rho B_{2n-1}} \right| \\
&= \left| \frac{\rho(\overline{B}_{2n} - \rho\overline{B}_{2n-1})(A_{2n}B_{2n-1} - A_{2n-1}B_{2n})}{(B_{2n} - \rho B_{2n-1})(|B_{2n}|^2 - \rho^2 |B_{2n-1}|^2)} \right| \\
&= \rho \prod_{v=1}^{2n} |c_v|^2 / |B_{2n}|^2 - \rho^2 |B_{2n-1}|^2 \\
&= \frac{\rho \prod_{v=1}^n |c_{2v-1}|^2}{1 - \rho^2 |B_{2n-1}/B_{2n}|^2 \prod_{v=1}^n |B_{2v}/c_{2v}B_{2v-2}|^2}
\end{aligned}$$

provided no $c_{2v}=0$. If $c_{2m}=0$, then $R_{2n}=0$ for all $n > m$, and the theorem is proved. In the case where $c_{2v} \neq 0$ for all $v \geq 1$ we apply Lemmas 1 and 2 and the fact that $|c_{2v-1}| < \rho$ to obtain

$$\begin{aligned}
R_{2n} &< \frac{n+1+r}{1+\rho} \rho^{2n+1} / \rho^{2n} \left(\prod_{v=1}^n \left(\frac{v+r+1}{v+r} \right) \right)^2 \\
&= \frac{\rho(n+1+r)}{(1+\rho)((n+1+r)/(1+r))^2} = \frac{\rho(1+r)^2}{(1+\rho)(n+1+r)}.
\end{aligned}$$

From this the theorem follows immediately.

We conclude with the following result.

THEOREM 2. *The continued fraction (1) converges at least in the wider sense if for all $n \geq 1$, $|c_{2n-1} \pm i| \geq \rho$, $|c_{2n}| \leq \rho$, where $0 < \rho < 1$.*

For the proof of this theorem it suffices to point out that

$$1 + \frac{c_2^2}{1 + \dots}$$

converges by Theorem 1.

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