SEMIGROUPS IN COMPACT GROUPS¹

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In the theory of one-parameter semigroups, the parameter family is usually an open semigroup of the real or complex numbers (under addition). Thus it is fundamental to investigate the structure of such semigroups. For a large class of these the structure is well known. These are the *angular semigroups* [3, Definition 7.6.1]; that is, those open semigroups which have the identity element 0 as a limit point. A complete discussion of angular semigroups of the real line and Euclidean 2-space will be found in [3, Chapter VII] and [4]. Furthermore, these methods of Hille and Zorn can be extended to a characterization of angular semigroups in Euclidean n -space E^n [12].

In this paper the structure of all open semigroups in any compact topological group is determined. We can then extend this result to a classification of the angular semigroups of any abelian topological group H which contains a compact open subgroup K . Coupling this with the results for Euclidean n -space, we obtain an essentially complete description of the angular semigroups of an arbitrary locally compact group.

For compact groups, the result is remarkably simple.

THEOREM I. *Let K be any compact group, and let S be a semigroup in K . Then S is necessarily a closed subgroup of K under either of the following conditions: (1) S is closed, (2) S is open.*

PROOF. The case where S is closed is already well known, and is contained in a theorem on topological semigroups due to several authors [2; 5; 6; 8; 9; 13]. The facts are these: S is a topological semigroup, or *mob*, and is compact. Then S must contain idempotents, and S is a topological group if and only if there is an identity element and there are no other idempotents in S . It is clear that these conditions are satisfied in the present case, and hence S is a group.

The case where S is open reduces to this. For, let $H = \bar{S}$; then H is a closed semigroup, and therefore a group. If the identity element 1 of K is in S and isolated in S , then it is isolated in K , and therefore K is discrete and finite. Either the standard algebraic result [15, Theorem 1, p. 3] or the above remarks then imply that S is a group. Otherwise, 1 is a limit point of S , and S is therefore angular. By [3, Theorem 7.6.2], we have $S = H^0$, where H^0 denotes the interior of H . But

Received by the editors April 25, 1955.

¹ Written under contract N7-onr-434, Task Order III, Office of Naval Research.

any subgroup of a topological group having a nonvoid interior is an open and closed subgroup, so that $S = H^0 = H$, q.e.d.

We may now appeal to the Zorn Category Theorem [3, Theorem 7.7.1] for the following consequence, which bears a striking resemblance to well known theorems of Banach [1, Theorem 1, p. 21] and Kuratowski [3, Theorem 1.8.1].

THEOREM II. *Let K be a compact group and let S be a semigroup in K which is of the second category at the identity 1 of K and which satisfies the condition of Baire. Then S is a subgroup of K which is both open and closed.*

PROOF. Zorn's theorem states that the interior S^0 of S is dense in S and that S^0 is the interior of $\bar{S}$. But S^0 is clearly a semigroup, and is not empty since S is not. Then Theorem I applies: $S^0 = S = \bar{S}$.

Slightly more general versions of the category theorem will be found in [7; 10].

A further result, reminiscent of the situation in locally compact groups, is the following immediate consequence of Theorem I.

THEOREM III. *In a compact group K , any semigroup S which is locally compact in its induced topology is a subgroup of K which is both open and closed.*

Now let H be an abelian topological group containing a compact open subgroup K , and let S be any open semigroup in H . If $S \cap K$ is not empty, then $S \cap K$ is an open subgroup of K , and hence of H , by Theorem I. On the other hand, if $S \cap K$ is empty, then S is not angular, since K is a neighborhood of the identity 1 of H . If $S \neq S \cap K$, let $x \in S, x \notin K$. Since S is a semigroup we have $x(S \cap K) \subset S$. This shows that if S^* is the image of S in $H/S \cap K$, then the complete inverse image of S^* is simply S again. Clearly S^* contains the identity of $H/S \cap K$. Conversely, if K_1 is any open subgroup of K , and if S^* is any semigroup in the (discrete) group H/K_1 which contains the identity, then the complete inverse image of S^* in H is an angular semigroup of H .

If G is any locally compact abelian group, then there exists a direct product decomposition $G = E^n \times H$ of G , where E^n is a Euclidean space of unique dimension n and where H is a group containing a compact open subgroup K [11; 14]. Then an angular semigroup of G is the direct product of angular semigroups in E^n and H . The above discussion, coupled with the results for Euclidean spaces, thus gives an essentially complete description of the angular semigroups of G .

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