THE COMPLEMENT OF A FINITELY GENERATED DIRECT SUMMAND OF AN ABELIAN GROUP[1]

P. M. COHN

1. In his recent monograph on abelian groups [1], Kaplansky raises the following question:

If F is a finitely generated abelian group and G, H are any abelian groups such that $F\oplus G\cong F\oplus H$, are G and H isomorphic?

We shall answer this question affirmatively. In the first place we can reduce the problem to the case where F is cyclic of infinite or prime power order. For suppose that the answer has been obtained in this case, and let F be any finitely generated abelian group. Then F is a direct sum of a finite number of cyclic groups, each of infinite or prime power order, say

$$F = F_1 \oplus F_2 \oplus \cdots \oplus F_k,$$

and

$$F_1 \oplus \cdots \oplus F_k \oplus G \cong F_1 \oplus \cdots \oplus F_k \oplus H.$$

We use induction on k. By the result for the cyclic case, we may cancel F_1 and obtain

$$F_2 \oplus \cdots \oplus F_k \oplus G \cong F_2 \oplus \cdots \oplus F_k \oplus H,$$

and therefore $G\cong H$, by induction.

2. It remains to deal with the cyclic case. By identifying $F\oplus G$ and $F\oplus H$ we may restate the assertion as follows: If E is an abelian group which may be written as a direct sum in two ways, $E=A\oplus G=B\oplus H$, where A and B are cyclic subgroups of E of the same order ∞ or p^n, then $G\cong H$.

3. We first dispose of the case where A and B are infinite. Let $G\cap H=D$, then $G/D\cong G/G\cap H\cong (G+H)/H\cong$subgroup of $E/H\cong B$. Thus G/D is infinite cyclic or 0. Similarly for H/D. If G/D and H/D are both zero, then $G=D=H$, so suppose that G/D is infinite cyclic. Choose $u\in G$ such that the residue-class $D+u$ generates G/D, and let U be the subgroup of G generated by u. Then $G=U\oplus D$, as is

Received by the editors June 16, 1955.

[1] Editor's note: A paper by E. A. Walker has been received whose contents are similar to those of the paper published here. Cf. also the paper of E. A. Walker, to be published in a later issue.

easily seen. Thus $E=A\oplus U\oplus D=B\oplus H$, and $D\subseteq H$. Taking quotients by D we obtain $A\oplus U\cong B\oplus H/D$, hence H/D is infinite cyclic; any representative in H of a generator mod D generates an infinite cyclic group V which satisfies $H=V\oplus D$. Thus $G=U\oplus D\cong V\oplus D=H$.

4. Now suppose that A and B are finite, of order p^n say, and are generated by a and b respectively. We show first that there exists an element u of order p^n such that no nonzero multiple of u belongs to G or H. If neither a nor b satisfies this condition, then since no nonzero multiple of a lies in G, we must have $p^{n-1}a\in H$, and similarly $p^{n-1}b\in G$. Put $u=a+b$, then $p^n u=0$, while $p^{n-1}u\equiv p^{n-1}a\not\equiv 0$ (mod G) and $p^{n-1}u\equiv p^{n-1}b\not\equiv 0$ (mod H). Thus there is always such an element u. Let U be the subgroup generated by u. Then from the definition, $U\cap G=0$, while $U+G/G\cong U/U\cap G\cong U$. Thus G has the index p^n in $U+G$ and the same index in E, which contains $U+G$. Therefore $E=U+G$, and in fact $E=U\oplus G$, because $U\cap G=0$. Similarly $E=U\oplus H$, and $G\cong E/U\cong H$.

Reference

1. I. Kaplansky, *Infinite abelian groups*, Ann Arbor, 1954.

The University,
Manchester, England