From Theorem 1 we see that the means $N(p)$ are equivalent with C_k for $k=1, 2, 3, 4$, but not for $k=5$ (and probably, not for any $k\geqq 5$).

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REARRANGEMENTS OF SERIES

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1. Introduction. Professor R. P. Agnew [1] and the author [2] have considered the metric space E of points $x\equiv(x_1, x_2, x_3, \cdots)$ where the complex $(x_1, x_2, x_3, \cdots)$ is a permutation of the positive integers and the distance between two points x and y is defined by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. \tag{1}$$

Starting with a given conditionally convergent series $\sum_{n=1}^{\infty} c_n$ of real terms, we associate it with the point $(1, 2, 3, \cdots)$ of the space E. Following Professor Agnew, we sometimes write the series $\sum c_n$ in the form $\sum c(n)$. With a given rearrangement of this series, say $c(n_1)+c(n_2)+c(n_3)+\cdots$, we associate in a unique manner the point $z=(n_1, n_2, n_3, \cdots)$ of the space E. For instance the rearranged series $c_2+c_1+c_4+c_3+\cdots$ is associated with the point $(2, 1, 4, 3, \cdots)$ of E. Conversely, with a given point $z=(n_1, n_2, \cdots)$ of E we associate the rearrangement $c(n_1)+c(n_2)+\cdots$ of the given series, and if the rearrangement converges to α we shall say that the series which corresponds to the point z converges to α.

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Let A denote the proper subset of E which contains each point of E which corresponds to a convergent rearrangement of $\sum c_n$. To each real number α there corresponds an infinite collection of rearrangements of $\sum c_n$ which converge to α. Hence to each real number α there corresponds an infinite subset of A which contains those points of A which correspond to rearrangements of $\sum c_n$ converging to α. The subset A may be endowed with the topology of E, and we may regard the given series as providing a function f which maps the space A onto the real number space.

Corresponding to different conditionally convergent series, the sets A in E whose points correspond to real numbers α in the sense described above are in general different. Even if we would start with a different rearrangement of the original series $\sum c_n$, we would perhaps get a different set A. In spite of this, these sets A exhibit some properties which are independent of the particular conditionally convergent series which give rise to them. Making free use of the notation and terminology from a book on topology by Professor R. Vaidyanathaswamy [**3**], we record here some properties of A and f. A referee is responsible for simplifications of the proofs.

2. **Properties of A.** In what follows, the set or space A is as defined above, and, for each x in A, $f(x)$ is the real number to which the rearrangement of $\sum c_n$ corresponding to x converges.

THEOREM 1. *The set A is a dense boundary subset of E.*

To prove that A is dense in E, we select any real number α and denote by $A(\alpha)$ the original of α so that $A(\alpha)$ consists of those points of A which correspond to series converging to α. The set $A(\alpha)$ may also be denoted by the symbol $f^{-1}(\alpha)$. We prove that $A(\alpha)$ is dense in E and since $A(\alpha)\subset A$ it follows that A is dense in E. Let p be a point of E so that $p=(a_1, a_2, \cdots)$ where $(a_1, a_2, \cdots)$ is a permutation of the positive integers, and let $\epsilon>0$. The sphere $S(p, \epsilon)$ with center at p and radius ϵ consists of the points $x=(x_1, x_2, \cdots)$ of E for which

$$d(p, x) = \sum \frac{1}{2^n} \frac{|a_n - x_n|}{1 + |a_n - x_n|} < \epsilon. \tag{2}$$

Let N be a positive integer such that $\sum_{k=N+1}^{\infty} 1/2^n<\epsilon$. Then each point q of E which has the form

$$q = (a_1, a_2, \cdots, a_N, y_{N+1}, y_{N+2}, \cdots), \tag{3}$$

where the first N elements of q are the same as the first n elements of p and the remaining elements of q constitute a permutation of the remaining positive integers, is such that

$$d(p, q) = \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{|a_n - y_n|}{1 + |a_n - y_n|} < \epsilon \tag{4}$$

and hence is such that $q \in S(p, \epsilon)$. Now it is possible to rearrange the series $c(y_{N+1}) + c(y_{N+2}) + \cdots$ in such a way that the rearranged series, say $c(y'_{N+1}) + c(y'_{N+2}) + \cdots$ converges to

$$\alpha - [c(a_1) + c(a_2) + \cdots + c(a_N)] \tag{5}$$

and therefore the series

$$c(a_1) + c(a_2) + \cdots + c(a_N) + c(y'_{N+1}) + c(y'_{N+2}) + \cdots \tag{6}$$

converges to α. Thus $S(p, \epsilon)$ contains a point $(a_1, \cdots, a_N, y'_{N+1}, y'_{N+2}, \cdots)$ of $A(\alpha)$ and it follows that $A(\alpha)$ and A are dense in E. To complete the proof of Theorem 1, it is sufficient to show that $S(p, \epsilon)$ contains a point of the set $A' = E - A$. That $S(p, \epsilon)$ contains a point of A' is a consequence of the fact that a permutation $y'_{N+1}, y'_{N+2}, \cdots$ of the integers $y_{N+1}, y_{N+2}, \cdots$ can be chosen in such a way that the series (6) is divergent and hence corresponds to a point in A'. The fact that $S(p, \epsilon)$ contains a point of A' is also a consequence of the theorem of Agnew [1] that A is of the first category in E. This completes the proof of Theorem 1. Our proof shows that $A(\alpha)$, A', and $A(\alpha)'$ are also dense boundary subsets of E.

3. **Properties of f.** We now study the function f which, by the procedure given above, maps the subspace A of E onto the real number space E_1.

THEOREM 2. *The map or function f is everywhere discontinuous over A.*

If x_0 is in A and $f(x_0) = \alpha_0$, then an easy modification of the proof of Theorem 1 shows that if $\delta > 0$ then the sphere $S(x_0, \delta)$ contains a point x of A for which $f(x) = \alpha_0 + 1$. The conclusion of Theorem 2 follows.

THEOREM 3. *The map f from A to E_1 is open in the sense that each nonempty subset of A which is open in A is mapped into a nonempty open set in E_1.*

Let A_1 be a nonempty subset of A which is open in A. Then there exist a point p in A and positive number ϵ such that A_1 contains $A \cap S(p, \epsilon)$. But, as the proof of Theorem 1 shows, there corresponds to each real α a point x of $A \cap S(p, \epsilon)$ for which $f(x)=\alpha$. This shows that the map of $A \cap S(p, \epsilon)$ is the whole space E_1 of real numbers and hence that the map of A_1 is E_1. Since E_1 is open, the conclusion of Theorem 3 follows.

We conclude with a theorem which employs the definition under which a map is said to be closed if the image of each closed set in the domain space is a closed set in the range space.

THEOREM 4. *The map f from A to E_1 is not closed.*

We prove this theorem by constructing a closed set in A whose image is the nonclosed set R of rational numbers in E_1. Let $r_1, r_2, r_3, \cdots$ be a sequence in which each rational number appears once and only once. Let $p_1=(x_1^{(1)}, x_2^{(1)}, \cdots)$ be a point of A such that the associated series $\sum c(x_n^{(1)})$ converges to r_1 so that $f(p_1)=r_1$. For each $k=2, 3, 4, \cdots$, let

$$p_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \cdots)$$

be a point of A which lies in the sphere $S(p_1, 2^{-k})$ with center at p_1 and radius 2^{-k} and which has an associated series $c(x^{(k)})+c(x^{(k)})+\cdots$ which converges to r_k so that $f(x_k)=r_k$. Since $2^{-k}\rightarrow 0$ as $k\rightarrow\infty$, the sequence $p_1, p_2, p_3, \cdots$ converges to its first element p_1. Let A_1 be the subset of A consisting of the points $p_1, p_2, p_3, \cdots$. The set A_1 is closed because it contains its one and only limit point p_1. However, the map of A_1 is not a closed set in the range space E_1 because this map is the nonclosed set R of rational numbers.

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