ON CONVERGENCE FIELDS OF NÖRLUND MEANS[1]

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A Nörlund mean $N(p)$ is defined by

$$(1)\quad \sigma_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu \quad \left(n \geqq N \text{ with } P_n = \sum_{\nu=0}^{n} p_\nu \neq 0 \text{ for } n \geqq N\right).$$

If $\sigma_n = s + o(1)$ when $n \to \infty$, the sequence $\{s_\nu\}$ (of complex numbers) is said to be limitable $N(p)$ to the value s. If $\sigma_n = o(1)$, we shall write $s_n \in o(N(p))$, denoting by $o(N(p))$ the set of all the sequences limitable $N(p)$ to zero. If $\sigma_n = a(1)$ when $n \to \infty$,[2] the sequence $\{s_\nu\}$ is said to be absolutely limitable $N(p)$, and we shall write $s_n \in a(N(p))$, denoting by $a(N(p))$ the set of all the sequences absolutely limitable $N(p)$.

A Nörlund mean is called regular if any convergent sequence $s_n \to s$ is transformed by (1) into a convergent sequence $\sigma_n \to s$. Necessary and sufficient conditions in terms of the sequence $\{p_n\}$ (of complex numbers) in order that (1) be regular are

$$(2)\qquad p_n = o(P_n) \qquad (n \to \infty),$$

and

$$(3)\qquad \sum_{\nu=0}^{n} |p_\nu| = O(P_n) \qquad (n \to \infty).$$

Similarly a Nörlund mean is called absolutely regular if any sequence $\{s_\nu\}$ with $s_n \to s$ and $s_n = a(1)$ is transformed by (1) into a sequence $\{\sigma_n\}$ with $\sigma_n \to s$ and $\sigma_n = a(1)$, and $N(p)$ is absolutely regular if and only if the conditions (2) and

$$(4)\qquad \sum_{n \geqq \mathrm{Max}\,(k, N+1)} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n-1-k}}{P_{n-1}} \right| \leqq K,$$

$$K \text{ independent of } k \ (k = 1, 2, \cdots),$$

are valid.[3]

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[2] Similarly to the symbols o and O the condition $a_n = a(b_n)$ means that there is a sequence $\{\alpha_n\}$ with $a_n = \alpha_n b_n$ $(n \geqq n_0)$ and $\{\alpha_n\}$ absolutely convergent, i.e. $\sum |\alpha_n - \alpha_{n+1}| < \infty$.

[3] Cf. Mears [4] and Knopp-Lorentz [2].

With any regular (respectively absolutely regular) Nörlund mean $N(p)$ there is associated a function $p(z)=\sum_0^\infty p_n z^n$ regular for $|z|<1$.[4]

Consider now two Nörlund means $N(p)$ and $N(r)$ where $N(p)$ is regular while $r(z)=\sum_0^\infty r_n z^n$ is regular for $|z|\leqq 1$ with $r(0)\neq 0$, $r(1)\neq 0$. Then $q(z)=r(z)p(z)$ belongs to a regular Nörlund mean $N(q)$. In this paper we shall ask for the relation between the convergence fields of the Nörlund means $N(p)$ and $N(r)$ on the one hand and the convergence field of the Nörlund mean $N(q)$ on the other hand. We shall show that there holds a certain additive relation (the following results are special cases of Theorem 1).

If $r(z)\neq 0$ for $|z|=1$ and $p(z)\neq 0$ for $|z|<1$, then $s_n\in o(N(q))$ if and only if $s_n=u_n+v_n$ where $u_n\in o(N(r))$ and $v_n\in o(N(p))$.

If we assume that $N(p)$ is absolutely regular, then the Nörlund mean $N(q)$ is absolutely regular. There holds a similar additive relation if o is replaced by a.

If $r(z)\neq 0$ for $|z|=1$ and $p(z)\neq 0$ for $|z|<1$, then $s_n\in a(N(q))$ if and only if $s_n=u_n+v_n$ where $u_n\in a(N(r))$ and $v_n\in a(N(p))$.

In the case $p(z)=1$ we are able to give explicitly all the sequences of $o(N(q))$ respectively $a(N(q))$. Suppose that $\{r_n\}$ satisfies the conditions of the first theorem above. Let α_i $(i=1, 2, \cdots, \kappa)$ be the zeros of $r(z)$ for $|z|<1$, α_i having the multiplicity $\gamma_i>0$. *Then $s_n\in o(N(r))$ if and only if $s_n=\sum_{i=1}^{\kappa}(1/\alpha_i^n)\sum_{j=0}^{\gamma_i-1}c_{ij}A_n^j+t_n$, where*

$$t_n=o(1),\qquad A_n^j=\binom{n+j}{n}\qquad\qquad (c_{ij}=\text{constant}),$$

and this result remains true with a in place of o.

When $r(z)$ is a polynomial, the o case of the last result recently was proved by Petersen.[5]

We remark that the condition $r(z)\neq 0$ for $|z|=1$ is essential as is easily seen[6] from the example $p(z)=1/(1-z)$, $r(z)=1+ze^{i\vartheta}-\pi<\vartheta<+\pi$. If $s_n\in o(N(p))=o(C_1)$, we have $s_n=o(n)$, and if $s_n\in o(N(r))$ we have $s_n=o(n)$, but if $s_n\in o(N(q))$ we have $s_n=o(n^2)$ while $s_n=o(n^\alpha)$ is not true for any $\alpha<2$ and all $s_n\in o(N(q))$.

We shall apply our results to the methods of Cesàro and Riesz (discontinuous), and we shall obtain the difference between the convergence fields of these methods with index $2k+1$ (k an integer).

1. **The main theorem.** Consider a sequence $\{p_n\}$ of complex numbers with the property

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[4] The regularity of $p(z)$ for $|z|<1$ follows easily from $P_{n+1}/P_n=1+o(1)$ because of (2). Cf. Hardy [1, p. 65].

[5] Petersen [5, Theorem 2.2, p. 452].

[6] By use of the Toeplitz theorem.

(5) $p_0 \neq 0$, and there exists a number N such that $\sum_{\nu=0}^{n} p_\nu \neq 0$ for $n \geqq N$.

Let $N_0 \geqq 0$ be the smallest number N with $\sum_{\nu=0}^{n} p_\nu \neq 0$ for $n \geqq N$. We define a sequence $\{P_n\}$ by $P_n = \sum_{\nu=0}^{n} p_\nu$ for $n \geqq N_0$, $P_n = 1$ for $0 \leqq n < N_0$.[7] If a sequence $\{p_n\}$ satisfies the conditions (5) and (2), the functions $\sum p_n z^n$ and $\sum P_n z^n$ are regular for $|z| < 1$ (because of (2), cf. footnote 3), and if $s_n \in o(N(p))$ or $s_n \in a(N(p))$, where $N(p)$ is the Nörlund mean (1), the series $\sum s_n z^n$ has a positive radius of convergence.[8]

From (5) and (2) we obtain the relations

$$p_{n+k} = o(P_n), \qquad P_{n+k}/P_n = 1 + o(1) \qquad (n \to \infty,\ k = 0, \pm 1, \pm 2, \cdots). \tag{6}$$

LEMMA 1. *If* $a_n = a(1)$, $b_n = a(1)$, *then* $a_n + b_n = a(1)$, $a_n b_n = a(1)$. *If* $a_n = a(1)$ *and* $a_n \to c \neq 0$, *then* $a_n^{-1} = a(1)$.

The proof of Lemma 1 is trivial.

If a sequence $\{p_n\}$ satisfies the conditions (5), (2), and

$$p_n = a(P_n) \qquad (n \to \infty), \tag{7}$$

then we obtain by use of Lemma 1 the relations

$$p_{n+k} = a(P_n), \qquad P_{n+k}/P_n = a(1) \qquad (n \to \infty,\ k = 0, \pm 1, \pm 2, \cdots). \tag{8}$$

LEMMA 2. *Consider a sequence subject to the conditions* (5) *and* (2). *Given a number* $\epsilon > 0$, *there exists a number* K_1 *with*

$$|P_{n+k}/P_n| \leqq K_1(1+\epsilon)^{|k|} \text{ for } n \geqq 0, \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{9}$$

K_1 *independent of* $k (P_{-1} = P_{-2} = \cdots = 0)$.

PROOF. From (6) we obtain the relation $1/(1+\epsilon) \leqq |P_{n+1}/P_n| \leqq 1+\epsilon$ for $n \geqq n_0(\epsilon)$, and the estimation (9) follows by an easy consideration.

LEMMA 3. *Consider a sequence* $\{p_n\}$ *subject to conditions* (5), (2), *and* (7). *Given a number* $\epsilon > 0$, *there exists a number* K_2 *with*

$$\sum_{n=0}^{\infty} \left| \frac{P_{n+k}}{P_n} - \frac{P_{n+1+k}}{P_{n+1}} \right| \leqq K_2 |k| (1+\epsilon)^{|k|} \text{ for } k = 0, \pm 1, \pm 2, \cdots, \tag{10}$$

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[7] If we change a finite number of the numbers P_n in (1), the convergence field $o(N(p))$ respectively $a(N(p))$ will not be altered. Therefore, in order to avoid some (formal) difficulties, we define the numbers P_n so that $P_n \neq 0$ for $n \geqq 0$.

[8] Because of $s(z) = \sum s_n z^n = (p(z))^{-1} \sum P_n \sigma_n z^n$, $p(0) \neq 0$. Cf. Hardy [1, p. 65].

K_2 *independent of* k $(P_{-1}=P_{-2}=\cdots=0)$.

PROOF. From

$$(P_{n+k}/P_n - P_{n+1+k}/P_{n+1}) = (P_{n+k}/P_{n+1+k} - P_n/P_{n+1})P_{n+1+k}/P_n$$

(where $n+k+1\geqq 0$ in case $k<0$) the estimation (10) follows because of (8) and (9).

LEMMA 4. *Consider a sequence* $\{p_n\}$ *subject to conditions* (5) *and* (2). *If* $\{d_n\}$ *is a sequence with* $\sum \zeta^n |d_n| < \infty$ *for some* $\zeta > 1$, *then*

$$(11)\quad \sum_{\nu=n}^{\infty} b_\nu d_{\nu-n} = \begin{cases} o(P_n) \\ P_n(d+o(1))\ (d=\sum d_n) \end{cases} \quad \text{if } b_n = \begin{cases} o(P_n) \\ P_n(1+o(1)), \end{cases}$$

and

$$(12)\qquad \sum_{\nu=0}^{n} b_\nu d_{n-\nu} = \begin{cases} o(P_n) \\ P_n(d+o(1)) \end{cases} \quad \text{if } b_n = \begin{cases} o(P_n) \\ P_n(1+o(1)). \end{cases}$$

PROOF. Because of (9) (with $1+\epsilon<\zeta$) we have for any $m=n+c$ $(c\geqq 0)$

$$\sum_{\nu=m}^{\infty}\left|\frac{P_\nu}{P_n}\right| |d_{\nu-n}| \leqq K_1 \sum_{\nu=m}^{\infty}(1+\epsilon)^{\nu-n}|d_{\nu-n}| \leqq K_1\sum_{\nu=c}^{\infty}\zeta^\nu |d_\nu|,$$

and from this estimation the relation (11) follows at once (observing relation (6)). In a similar manner we obtain the estimation (12) (notice that $|d_n/P_n| \leqq K_1(1+\epsilon)^n|d_n| = o(1)$).

LEMMA 5. *Consider a sequence* $\{p_n\}$ *subject to conditions* (5), (2), *and* (7). *If* $\{d_n\}$ *is a sequence with* $\sum\zeta^n|d_n|<\infty$ *for some* $\zeta>1$, *then*

$$(13)\qquad \sum_{\nu=n}^{\infty} b_\nu d_{\nu-n} = a(P_n) \qquad \text{if } b_n = a(P_n),$$

and

$$(14)\qquad \sum_{\nu=0}^{n} b_\nu d_{n-\nu} = a(P_n) \qquad \text{if } b_n = a(P_n).$$

PROOF. Consider a sequence $\delta_n = a(1)$. From

$$\begin{aligned}&\delta_{n+k}P_{n+k}/P_n - \delta_{n+k+1}P_{n+k+1}/P_{n+1} \\ &\qquad = \delta_{n+k}(P_{n+k}/P_n - P_{n+k+1}/P_{n+1}) + P_{n+k+1}/P_{n+1}(\delta_{n+k}-\delta_{n+k+1}) \\ &\qquad\qquad (\delta_\nu = 0 \text{ and } P_\nu = 0 \text{ in case } \nu<0)\end{aligned}$$

we obtain (because of (9) and (10)) the relation

$$(15)\quad \sum_{n=0}^{\infty}\left|\delta_{n+k}\frac{P_{n+k}}{P_n}-\delta_{n+1+k}\frac{P_{n+1+k}}{P_{n+1}}\right| \leqq K_3K_2|k|(1+\epsilon)^{|k|}$$
$$+K_1(1+\epsilon)^{|k|}K_4 \leqq K_5(|k|+1)(1+\epsilon)^{|k|},$$

$(k=0, \pm 1, \pm 2, \cdots)$ with K_5 independent of k.

Putting $b_n=\delta_n P_n$ and $\beta_n=\sum_{\nu=n}^{\infty}(b_\nu/P_n)d_{\nu-n}$ we have

$$\beta_n-\beta_{n+1}=\sum_{\nu=0}^{\infty}\left(\delta_{n+\nu}\frac{P_{n+\nu}}{P_n}-\delta_{n+1+\nu}\frac{P_{n+1+\nu}}{P_{n+1}}\right)d_\nu,$$

and by use of (15) we obtain

$$\sum_{n=0}^{\infty}|\beta_n-\beta_{n+1}| \leqq K_5\sum_{\nu=0}^{\infty}(\nu+1)(1+\epsilon)^\nu|d_\nu| = K_6 \quad (1+\epsilon<\zeta),$$

which proves (13).

By use of (15) we obtain (14) in a similar manner.

Lemma 6. *Let $r(z)=\sum r_n z^n$ be convergent for $|z|\leqq\rho$, $\rho>1$, $r(0)\neq 0$, $r(1)\neq 0$, and define $q(z)$ by $q(z)=r(z)p(z)$ $(q(z)=\sum q_n z^n$, $p(z)=\sum p_n z^n)$.*

(i) *If $\{p_n\}$ satisfies conditions* (5) *and* (2), *then $\{q_n\}$ satisfies* (5) *and* (2), *and the relation*

$$(16)\qquad Q_n/P_n = r(1)+o(1)$$

is true. (*Because of* (5) *the sequence $\{Q_n\}$ may be generated by $\{q_n\}$ in the same way as is $\{P_n\}$ by $\{p_n\}$.*)

(ii) *If $\{p_n\}$ satisfies conditions* (5), (2), *and* (7), *then $\{q_n\}$ satisfies* (5), (2), *and* (7), *and the relation*

$$(17)\qquad Q_n/P_n = a(1)$$

is true.

Proof. Putting $Q_n^*=\sum_{\nu=0}^{n} q_\nu$, $P_n^*=\sum_{\nu=0}^{n} p_\nu$ $(P_n^*=P_n$ for $n\geqq N_0)$ we have the relation

$$(18)\qquad Q_n^*=\sum_{\nu=0}^{n}P_\nu^* r_{n-\nu}.$$

If $\{p_n\}$ is subject to conditions (5) and (2), we obtain from (18) by (12) the relation

$$(19)\qquad Q_n^*/P_n^* = r(1)+o(1) \qquad (n\to\infty),$$

and therefore $\{q_n\}$ satisfies the condition (5) (notice that $q_0=r_0p_0$). Considering $q_n/Q_n=q_n/P_n\cdot P_n/Q_n$ we obtain the statements of the lemma from the relation $q_n = \sum_{\nu=0}^{n} p_\nu r_{n-\nu}$, by (19), Lemma 4, Lemma 5, and Lemma 1.

LEMMA 7. *Let $\hat{p}(z) = \sum \hat{p}_n z^n$ be convergent for $|z| < 1$, and define $p(z)$ by $p(z) = (1 - z/\alpha)^k \hat{p}(z)$ with $0 < |\alpha| < 1$ and $k > 0$ an integer.*

(i) *If $\{p_n\}$ satisfies conditions* (5) *and* (2), *then $\{\hat{p}_n\}$ satisfies* (5) *and* (2), *and the relation*

$$\widehat{P}_n/P_n = (1 - 1/\alpha)^{-k} + o(1) \tag{20}$$

is true. (*For the definition of $\widehat{P}_n$ cf. Lemma* 6 (i).)

(ii) *If $\{p_n\}$ satisfies conditions* (5), (2), *and* (7), *then $\{\hat{p}_n\}$ satisfies* (5), (2), *and* (7), *and the relation*

$$\widehat{P}_n/P_n = o(1) \tag{21}$$

is true.

PROOF. Suppose first $k = 1$. Putting $P_n^* = \sum_{\nu=0}^{n} p_\nu$ ($P_n^* = P_n$ for $n \geqq N_0$) and $\widehat{P}_n^* = \sum_{\nu=0}^{n} \hat{p}_\nu$, we obtain from $\hat{p}(z) = (1 - z/\alpha)^{-1} p(z)$ the relations

$$\hat{p}_n = \frac{1}{\alpha^n} \sum_{\nu=0}^{n} p_\nu \alpha^\nu = - \sum_{\nu=n+1}^{\infty} p_\nu \alpha^{\nu-n} \qquad \text{(because of } p(\alpha) = 0\text{)}, \tag{22}$$

and

$$\widehat{P}_n^* = \frac{1}{\alpha^n} \sum_{\nu=0}^{n} P_\nu^* \alpha^\nu = - \sum_{\nu=n+1}^{\infty} P_\nu \alpha^{\nu-n} \qquad \text{for } n \geqq N_0. \tag{23}$$

If $\{p_n\}$ is subject to conditions (5) and (2), then we obtain from (23) by (11) (putting $d_n = \alpha^{u+1}$) the relation

$$\widehat{P}_n^*/P_n = (1 - 1/\alpha)^{-1} + o(1), \tag{24}$$

hence (because of $\hat{p}_0 = p_0$) $\{\hat{p}_n\}$ satisfies the relation (5). Considering $\hat{p}_n/\widehat{P}_n = \hat{p}_n/P_n \cdot P_n/\widehat{P}_n$ we derive the statements of the lemma from (22) and (24) by Lemma 4, Lemma 5 (putting $d_n = \alpha^{n+1}$), and Lemma 1.

For $k > 1$ the result follows by induction.

LEMMA 8. *Consider a function $p(z) = \sum p_n z^n$, regular for $|z| < 1$, and a number α with $0 < |\alpha| < 1$. Suppose that $p(z)$ has a root of multiplicity $\lambda \geqq 0$ for $z = \alpha$ (i.e. $p(z) = (1 - z/\alpha)^\lambda h(z)$ where $h(z) = \sum h_n z^n$ is regular for $|z| < 1$ and $h(\alpha) \neq 0$), and define $q(z)$ by $q(z) = (1 - z/\alpha) p(z)$.*

(i) *If $\{p_n\}$ satisfies conditions* (5) *and* (2), *then $\{q_n\}$ satisfies conditions* (5) *and* (2), *and $s_n \in o(N(q))$ if and only if*

$$s_n = t_n + c\frac{A_n^{\lambda}}{\alpha^n}, \tag{25}$$
$$t_n \in o(N(p)), \left(A_n^{\lambda} = \binom{n+\lambda}{n}, \ c = \text{constant}\right).$$

(ii) *If* $\{p_n\}$ *satisfies conditions* (5), (2), *and* (7), *then* $\{q_n\}$ *satisfies conditions* (5), (2), *and* (7), *and* $s_n \in a(N(q))$ *if and only if*

$$s_n = t_n + c(A_n^{\lambda}/\alpha^n), \qquad t_n \in a(N(p)). \tag{26}$$

PROOF. Suppose that $\{p_n\}$ is subject to conditions (5) and (2). By Lemma 6 the sequence $\{q_n\}$ satisfies conditions (5) and (2). Consider a sequence $s_n \in o(N(q))$. The series $s(z) = \sum s_n z^n$ has a positive radius of convergence, and we define a function $t(z) = \sum t_n z^n$ by $t(z) = s(z) - c(1-z/\alpha)^{-(\lambda+1)}$ (c = constant). Putting $q(z)s(z) = a(z) = \sum a_n z^n$ ($a_n = o(Q_n)$ because of $s_n \in o(N(q))$, so that $a(z)$ is regular for $|z| < 1$), we have

$$p(z)t(z) = \frac{q(z)}{1 - z/\alpha}(s(z) - c(1 - z/\alpha)^{-(\lambda+1)}) = \frac{1}{1 - z/\alpha}(a(z) - ch(z))$$
$$= \frac{1}{1 - z/\alpha}\sum b_n z^n \qquad (b_n = a_n - ch_n).$$

If we put $c = a(\alpha)/h(\alpha)$, we have

$$p(z)t(z) = -\sum_{n=0}^{\infty} z^n \sum_{\nu=n+1}^{\infty} b_\nu \alpha^{\nu-n} \qquad \left(\text{because of } \sum b_n\alpha^n = 0\right). \tag{27}$$

From (16) and $a_n = o(Q_n)$ we obtain $a_n = o(P_n)$, and by use of Lemma 7 we have

$$h_n = o(P_n), \tag{28}$$

so that $b_n = o(P_n)$, and from this estimation we get

$$\sum_{\nu=n+1}^{\infty} b_\nu \alpha^{\nu-n} = o(P_n) \tag{29}$$

by Lemma 4. As is seen from (27), the estimation (29) means that $t_n \in o(N(p))$.

It remains to show that any sequence (25) is contained in $o(N(q))$. If $t_n \in o(N(p))$ we have $p(z)t(z) = b(z) = \sum b_n z^n$ with $b_n = o(P_n)$. Putting $q(z)t(z) = a(z) = \sum a_n z^n = (1 - z/\alpha)b(z)$, we obtain $a_n = o(P_n)$ (by use of (12)), and because of $Q_n = P_n(1-1/\alpha+o(1))$ (by Lemma 6) we have $a_n = o(Q_n)$ so that $t_n \in o(N(q))$.

Finally we have $A_n^\lambda/\alpha^n \in o(N(q_n))$ because of $q(z)\cdot\sum_{n=0}^{\infty}(A_n^\lambda/\alpha^n)z^n = h(z)$, (28), and (16).

Writing a instead of o, the proof of (ii) runs in exactly the same lines as the proof of (i).

LEMMA 9. *Consider a function $p(z)=\sum p_n z^n$, regular for $|z|<1$, and a function $r(z)=\sum r_n z^n$, regular for $|z|\leqq 1$ with $r(z)\neq 0$ for $|z|\leqq 1$. Define $q(z)$ by $q(z)=r(z)p(z)$.*

(i) *If $\{p_n\}$ satisfies conditions* (5) *and* (2), *then $\{q_n\}$ satisfies conditions* (5) *and* (2), *and we have $o(N(p))=o(N(q))$.*

(ii) *If $\{p_n\}$ satisfies conditions* (5), (2), *and* (7), *then $\{q_n\}$ satisfies conditions* (5), (2), *and* (7), *and we have $a(N(p))=a(N(q))$.*

PROOF. Suppose that $\{p_n\}$ is subject to conditions (5) and (2). By Lemma 6 the sequence $\{q_n\}$ satisfies conditions (5) and (2). Consider a sequence $s_n \in o(N(p))$. Putting

$$p(z)s(z) = a(z) = \sum a_n z^n (s(z) = \sum s_n z^n),$$

we have $a_n = o(P_n)$. Putting $q(z)s(z)=b(z)=\sum b_n z^n = r(z)a(z)$ we have the relation

$$b_n = \sum_{\nu=0}^{n} a_\nu r_{n-\nu}. \tag{30}$$

Using Lemma 4, we derive from (30) the relation $b_n = o(P_n)$, and by Lemma 6 we obtain $b_n = o(Q_n)$. Hence we have $s_n \in o(N(q))$ and therefore $o(N(p))\subseteq o(N(q))$. From $p(z)=r(z)^{-1}q(z)$ where $r(z)^{-1}$ satisfies the same conditions as does $r(z)$, we obtain the relation $o(N(q))\subseteq o(N(p))$, and this yields the result desired.

The proof of (ii) runs in exactly the same lines as the preceeding proof of (i).

The following theorem is obtained by a combination of Lemma 8 and Lemma 9.

THEOREM 1. *Consider a function $r(z)=\sum r_n z^n$, regular for $|z|\leqq 1$, $r(z)\neq 0$ for $z=0$ and $|z|=1$, and having inside the unit circle the roots $\alpha_1 \cdots \alpha_\kappa$ with the multiplicities $\gamma_1 \cdots \gamma_\kappa$*

$$\left(i.e.\ r(z) = \prod_{i=1}^{\kappa}(1-z/\alpha_i)^{\gamma_i} r_1(z) \text{ where } r_1(z)\neq 0 \text{ for } |z|\leqq 1, \gamma_i > 0\right).$$

Let $p(z)=\sum p_n z^n$ be a function regular for $|z|<1$ and suppose *that* α_i (*i.e. the ith root of* $r(z)$) *is a root of* $p(z)$ *with multiplicity* $\lambda_i \geqq 0$ ($i=1, 2, \cdots, \kappa$). *Define* $q(z)$ *by* $q(z)=r(z)p(z)$.

(i) *If* $\{p_n\}$ *satisfies conditions* (5) *and* (2), *then* $s_n \in o(N(q))$ *if and only if*

$$(31)\quad \begin{aligned} s_n &= t_n + \sum_{i=1}^{\kappa} \frac{1}{\alpha_i^n} \sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} c_{ij} A_n^j, \\ & t_n \in o(N(p)), \left(A_n^j = \binom{n+j}{n}, \quad c_{ij} = \text{constant}\right). \end{aligned}$$

(ii) *If* $\{p_n\}$ *satisfies conditions* (5), (2), *and* (7), *then* $s_n \in a(N(q))$ *if and only if* (31) *holds with* $t_n \in a(N(p))$.

In case (i) *the sequence* $\{q_n\}$ *satisfies conditions* (5) *and* (2), *and in case* (ii) *the sequence* $\{q_n\}$ *satisfies* (5), (2), *and* (7).

Proof. Considering the functions

$$(1 - z/\alpha_1)p(z), (1 - z/\alpha_1)^2 p(z), \quad \cdots, (1 - z/\alpha_1)^{\gamma_1} p(z),$$

$$(1 - z/\alpha_2)(1 - z/\alpha_1)^{\gamma_1} p(z), \cdots, \prod_{i=1}^{\kappa} (1 - z/\alpha_i)^{\gamma_i} p(z) = p^*(z),$$

we obtain by a repeated application of Lemma 8 the result that the convergence field of $p^*(z)$ is given by (31) (respectively by (31) with $t_n \in a(N(p))$ in case (ii)). By Lemma 9 the convergence fields of the Nörlund means belonging to $p^*(z)$ and $r_1(z)p^*(z)=q(z)$ are not different.

Finally we shall show that the theorems stated in the introduction of this paper are special cases of Theorem 1. These theorems deal with regular (absolutely regular) Nörlund means while the weaker conditions (5) and (2) ((5), (2), and (7)) imply that the $N(p)$-transformation of any sequence $\{s_n\}$ with $s_n=0$ for all large n tends to zero (tends to zero and is absolutely convergent).

Given a regular (absolutely regular) Nörlund mean[9] $N(p)$ and a sequence $\{r_n\}$ such that $r(z)=\sum r_n z^n$ is convergent for $|z| \leqq \zeta$, $\zeta > 1$, $r(0) \neq 0$ and $r(1) \neq 0$, then by $q(z)=r(z)p(z)$ a regular (absolutely regular) Nörlund mean $N(q)$ is defined. In fact, suppose that $s_n=o(1)$ ($s_n=o(1)$ and $s_n=a(1)$).[10] Putting

$$\sigma_n = (1/P_n) \sum_{\nu=0}^{n} p_{n-\nu} s_\nu$$

[9] And $\{p_n\}$ subject to condition (5).

[10] Obviously it is sufficient to consider only sequences tending to zero.

we have

$$\sum_{\nu=0}^{n} q_{n-\nu}s_\nu = \sum_{\nu=0}^{n} r_{n-\nu}P_\nu\sigma_\nu = o(P_n)(= o(P_n) \quad \text{and} \quad = a(P_n))$$

by Lemma 4 (Lemma 4 and Lemma 5) because of $\sigma_n = o(1)$ ($\sigma_n = o(1)$ and $\sigma_n = a(1)$) and from (16) ((16) and (17)) we obtain the relation $\sum_{\nu=0}^{n} q_{n-\nu}s_\nu = o(Q_n)$ ($=o(Q_n)$ and $=a(Q_n)$) q.e.d.

If we put $p(z)=1$, the last theorem of the introduction follows at once from Theorem 1, and combining this result again with Theorem 1 we obtain the first and the second theorem of the introduction (the second term in (31) (with $\lambda_i=0$) plus all sequences tending to zero represents all sequences belonging to $o(N(r))$, similarly in the absolute case).

2. **An application.** As an application of Theorem 1 we shall investigate the convergence fields of the Cesàro means $C_k = N(p)$ with

$$p_n = \binom{n+k-1}{n} \qquad (k>0),$$

and the discontinuous Riesz means $R_n^k = N(q)$ with $q_n = (n+1)^k - n^k$ ($k>0$). We shall assume that k is an integer. The sequences $\{p_n\}$ and $\{q_n\}$ thus defined satisfy conditions (2), (5), and (7), and C_k is connected with the function $p(z) = 1/(1-z)^k$ while R_n^k is connected with the function

$$(32) \quad q(z) = (1-z)\sum_{n=0}^{\infty}(n+1)^k z^n = (1-z)\left(\frac{d}{dz}z\right)^k \frac{1}{1-z} = \frac{P_{k-1}(z)}{(1-z)^k},$$

$P_{k-1}(z)$ being a polynomial of degree $k-1$ (this follows easily by induction from $P_0=1$, $P_1 = 1+z$, $P_2 = 1+4z+z^2$).

From (32) we obtain the relation $q(z) = P_{k-1}(z)p(z)$, and therefore, in order to apply Theorem 1, we have to investigate the distribution of the roots of $P_{k-1}(z)$.[11]

LEMMA 10. *The polynomials $P_k(z)$ ($k=0, 1, \cdots$) are reciprocals, i.e. $P_k(z) = z^k P_k(1/z)$.*

PROOF. Starting with $P_0(z)=1$, we proceed by induction. Assuming that $P_{k-1}(z) = z^{k-1}P_{k-1}(1/z)$ ($k \geqq 1$), we have for $k \geqq 1$ the relation[12]

[11] Some properties of the function $\sum(n+1)^k z^n$ (including the distribution of the roots) have been investigated by Lawden [3]. Here we shall give a short proof of those properties of the roots of $P_{k-1}(z)$ we need for our purposes.

[12] Because of the (formal) relation $(d/d\zeta)f(\zeta)|_{\zeta=1/a} = -x^2(d/dx)f(1/x)|_{x=a}$.

$$P_k\left(\frac{1}{z}\right) = \left(1 - \frac{1}{z}\right)^{k+2} \frac{d}{d\zeta} \frac{\zeta P_{k-1}(\zeta)}{(1-\zeta)^{k+1}}\bigg|_{\zeta=1/z}$$
$$= -z^2\left(1 - \frac{1}{z}\right)^{k+2} \frac{d}{dz} \frac{(1/z)P_{k-1}(1/z)}{(1-1/z)^{k+1}}$$
$$= -z^2\left(1 - \frac{1}{z}\right)^{k+2} \frac{d}{dz} \frac{zP_{k-1}(z)}{(z-1)^{k+1}}$$
$$= -z^2\left(1 - \frac{1}{z}\right)^{k+2} (-1)^{k+1} \frac{P_k(z)}{(1-z)^{k+2}} = \frac{1}{z^k} P_k(z).$$

LEMMA 11. *All roots of $P_k(z)$ ($k \geqq 0$) are located on the negative real axis. Exactly k different roots of $P_{2k}(z)$ and $P_{2k+1}(z)$ are contained in any of the intervals $-\infty < z < -1$, $-1 < z < 0$, and we have $P_{2k}(-1) \neq 0$, $P_{2k+1}(-1) = 0$.*

PROOF. Putting $f_k(z) = P_k(z)/(1-z)^{k+2}$ we have $f_{k+1}(z) = (zf_k(z))'$ ($z \neq 1$), and by integration the formula

$$\int_a^b f_{k+1}(x)dx = bf_k(b) - af_k(a) \qquad (x \text{ real}, a < b < +1, k \geqq 0). \tag{33}$$

From (33) we derive at once the following properties of $P_k(z)$:

(i) Given two real numbers α and β with $\alpha < \beta < 0$ and $P_k(\alpha) = P_k(\beta) = 0$, there exists a real number γ with $\alpha < \gamma < \beta$ such that $P_{k+1}(\gamma) = 0$ (put $a = \alpha$, $b = \beta$ in (33)).

(ii) Given a real number $\alpha < 0$ with $P_k(\alpha) = 0$, there exists a real number γ with $\alpha < \gamma < 0$ such that $P_{k+1}(\gamma) = 0$ (put $b = 0$, $a = \alpha$ in (33)).

Starting with $P_1(z) = 1 + z$ and proceeding by induction, we obtain the result desired by an easy consideration from (i), (ii), and Lemma[13] 10 (observe that P_k has at most k different roots).

Using Lemma 11, we derive from Theorem 1 the following

THEOREM 2. *A sequence $\{s_n\}$ is an element of $o(R_n^{2k+1})$ ($k \geqq 0$ an integer) ($a(R_n^{2k+1})$) if and only if*

$$s_n = t_n + \sum_{i=1}^{k} \frac{c_i}{\alpha_i^n}, \qquad t_n \in o(C_{2k+1})(t_n \in a(C_{2k+1}))$$

where c_i ($i = 1, 2, \cdots, k$) is constant and the numbers α_i are the roots of $P_{2k}(z)$ located inside the unit circle.

It may be of some interest to have an estimation for the modulus of the smallest root of $P_k(z)$.

[13] By Lemma 10 we have $P_{2k+1}(-1) = 0$ and $P_k(1/z_0) = 0$ if $P_k(z_0) = 0$.

LEMMA 12. *Let $\alpha_0(k)$ denote the root with smallest modulus of $P_k(z)$. Then we have the estimation*

$$\frac{1+o(1)}{2^{k+1}} \leqq |\alpha_0(k)| \leqq k\frac{1+o(1)}{2^{k+1}} \tag{34}$$
$$\left(\text{with } 1+o(1) = \frac{1}{1-(k+2)/2^{k+1}}\right).$$

PROOF. From (32) we have $P_k(0)=1$ and therefore

$$P_k(z) = \prod_{i=1}^{k}(z-\alpha_i)$$

($k \geqq 1$, observe Lemma 10 and $a_0 = a_n = 1$ (because of Lemma 11)). Considering the logarithmic derivative of this identity we obtain

$$P_k'(z)/P_k(z) = -\sum_{n=0}^{\infty} z^n \sum_{i=1}^{k} 1/\alpha_j^{n+1} \qquad |z| \text{ being small.}$$

From $P_k'(0)/P_k(0) = \sum_{i=1}^{k} 1/|\alpha_i|, 1/|\alpha_0(k)| \leqq \sum_{i=1}^{k} 1/|\alpha_i| \leqq k/|\alpha_0(k)|$ and $P_k'(0) = 2^{k+1}-(k+2)$ (this relation follows by an easy calculation from (32)) we obtain (34).

Finally we consider the Nörlund mean $N(\hat{p})$ defined by $\hat{p}_0 = k^k$, $\hat{p}_n = (n+k)^k-(n-1+k)^k$, $n \geqq 1$ ($k>0$ an integer). We obtain this mean from R_n^k omitting the first $(k-1)$ elements of the sequence $\{p_n\}$.[14] Similarly to (32) we have for $\hat{p}(z)$ the relation

$$\hat{p}(z) = (1-z)\sum_{n=0}^{\infty}(n+k)^k z^n = (1-z)\frac{1}{z^{k-1}}\left(\frac{d}{dz}z\right)^k \frac{z^{k-1}}{1-z} \tag{35}$$
$$= \frac{Q_k(z)}{(1-z)^k}.$$

We have $Q_1(z)=1$, $Q_2(z)=z^2-3z+4$, $Q_3(z)=-8z^3+31z^2-44z+27$, $Q_4(z)=81z^4-389z^3+731z^2-655z+256$, $Q_5(z)=-1024z^5+5901z^4-13934z^3+17026z^2-10974z+3125$.

By numerical calculation we obtain the following roots of smallest modulus

Q_2: $z_0 = 1.5+i1.32\cdots$, $(=3/2+i7^{1/2}/2)$, $|z_0|^2 = 4$,

Q_3: $z_0 = 0.9710+i0.8961+\epsilon$, $|\epsilon| < 3.10^{-4}$, $|z_0|^2 = 1.74\cdots$,

Q_4: $z_0 = 0.80299+i0.71437+\epsilon$, $|\epsilon| < 10^{-5}$, $|z_0|^2 = 1.155\cdots$,

Q_5: $z_0 = 0.72034+i0.60896+\epsilon$, $|\epsilon| < 5.10^{-5}$, $|z_0|^2 = 0.89\cdots$.

[14] A Nörlund mean of this kind has been considered by Riesz [6].

From Theorem 1 we see that the means $N(p)$ are equivalent with C_k for $k=1, 2, 3, 4$, but not for $k=5$ (and probably, not for any $k \geqq 5$).

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REARRANGEMENTS OF SERIES

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1. Introduction. Professor R. P. Agnew [1] and the author [2] have considered the metric space E of points $x \equiv (x_1, x_2, x_3, \cdots)$ where the complex $(x_1, x_2, x_3, \cdots)$ is a permutation of the positive integers and the distance between two points x and y is defined by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. \tag{1}$$

Starting with a given conditionally convergent series $\sum_{n=1}^{\infty} c_n$ of real terms, we associate it with the point $(1, 2, 3, \cdots)$ of the space E. Following Professor Agnew, we sometimes write the series $\sum c_n$ in the form $\sum c(n)$. With a given rearrangement of this series, say $c(n_1)+c(n_2)+c(n_3)+ \cdots$, we associate in a unique manner the point $z=(n_1, n_2, n_3, \cdots)$ of the space E. For instance the rearranged series $c_2+c_1+c_4+c_3+ \cdots$ is associated with the point $(2, 1, 4, 3, \cdots)$ of E. Conversely, with a given point $z=(n_1, n_2, \cdots)$ of E we associate the rearrangement $c(n_1)+c(n_2)+ \cdots$ of the given series, and if the rearrangement converges to α we shall say that the series which corresponds to the point z converges to α.

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