THE PONTRJAGIN RING FOR CERTAIN LOOP SPACES

ARTHUR H. COPELAND, JR.[1]

Suppose A is a principal ideal ring and G is a graded A-module. Then an algebra $T^*(G)$ is described. This is called the ***tensor-torsion ring of G***. If X is a 1-connected space of Lusternik-Schnirelmann strong category two and Ω the space of loops on X then $H_*(\Omega; A) = T^*(\sum_{n=2}^{\infty} H_n(X, A))$.

1. The tensor-torsion ring. Let C be a free A-module with endomorphisms d, k such that

$$d^2 = 0, \qquad dk = -kd.$$

Let $T(C)$ designate the tensor ring of C [2]. That is, $T(C)^+ = \sum_{n=0}^{\infty} C^{(n)}$ when $C^{(n)}$ is defined by

$$C^{(0)} = A,$$
$$C^{(n)} = C \otimes C^{(n-1)} \qquad \text{for } n > 0$$

and products on $T(C)$ are induced by: if $c \in C^{(n)}$, $c' \in C^{(m)}$ then $c \cdot c' = c \otimes c' \in C^{(n+m)}$ (where $C^{(n)} \otimes C^{(m)}$ is identified with $C^{(n+m)}$ via the obvious rearrangement of parentheses). The tensor products are taken relative to A.

Next, define the endomorphisms d_n: $C^{(n)} \to C^{(n)}$ by:

$$d_0(a) = 0, \qquad a \in A = C^{(0)},$$
$$d_1(c) = dc, \qquad c \in C = C^{(1)},$$
$$d_n(c \otimes c') = dc \otimes c' + kc \otimes d_{n-1}c', \qquad c \in C^{(1)}, c' \in C^{(n-1)}.$$

These are extended linearly to give D: $T(C) \to T(C)$. Note that D is a ring endomorphism with $D^2 = 0$. Set

$$Z = Z(T(C)) = \operatorname{Ker} D,$$
$$B = B(T(C)) = \operatorname{Im} D.$$

It is easily seen that Z is a subring of $T(C)$ and B is a 2-ideal in Z. Thus $Z/B = H(T(C))$ is a ring.

We shall say that a group is A-free when it is a free A-module.

Received by the editors January 20, 1955 and, in revised form, June 16, 1955.

[1] A portion of this paper is taken from the author's doctoral dissertation, Massachusetts Institute of Technology, 1954.

PROPOSITION 1. *If $G=\sum_{n=0}^{\infty} G^n$ is a graded A-module then there is an A-free chain group $C=\sum_{n=0}^{\infty} C^n$ such that $G^n \approx H^n(C)$.*

PROOF. Let Z^n be a free A-module with generators in 1-1 correspondence with the elements of G^n. This correspondence may be uniquely extended to a homomorphism θ^n: $Z^n \to G^n$. We define: $B^n = \ker \theta^n$, W^{n+1} $(n \geqq 0)$ a group isomorphic to B^n with $\bar{d}$: $W^{n+1} \to B^n$ the isomorphism, and $W^0 = (0)$. Set $C^n = Z^n \oplus W^n$ and $d(z, w) = (\bar{d}w, 0)$ for $z \in Z^n$, $w \in W^n$, $n \geqq 0$. Then $d^2 = 0$ and $Z^n/B^n = H^n(C) \approx G^n$.

The endomorphism k on C is now taken to be $kc = (-1)^n c$ for $c \in C^n$. In terms of this notation:

We set $T^*(G) = H(T(C))$. $T^*(G)$ is called the tensor-torsion ring (or algebra) of G.

The excuse for calling $T^*(G)$ the tensor-torsion ring of G is that the additive structure of $T^*(G)$ is $\sum_{n=0}^{\infty} H(C^{(n)})$ which by the Kunneth formula contains, as subgroups, isomorphic images of all combinations of tensor and torsion products of G with itself. The multiplication in $T^*(G)$ is essentially the tensor product.

PROPOSITION 2. *If C, C' are A-free chain groups and homomorphisms θ^n: $H_n(C) \to H_n(C')$ are given for each $n \geqq 0$, then there is a chain map Θ: $C \to C'$ which induces the homomorphisms θ^n.*

PROOF. Let d, d' be the boundary homomorphisms on C, C' respectively and ϕ: $Z^n \to H_n(C)$, ϕ': $Z'^n \to H_n(C')$ the natural projections from the cycle groups. Since Z^n is A-free, we may pick a basis $A^n \subset Z^n$. If $\alpha \in A^n$, let $f^n\alpha$ be an element of $(\phi')^{-1}\theta\phi\alpha$. Extend f^n to g^n: $Z^n \to Z'^n$. But Z^n, Z'^n are direct summands of C^n, C'^n; let W^n be a subgroup such that $C^n = Z^n \oplus W^n$ and define W'^n similarly. Define $h^n(w) = (d' | W'^n)^{-1} \circ g^{n-1} \circ d(w)$ for $w \in W^n$. Extend $\{g^n, h^n\}$ to Θ: $C \to C'$. It is immediate from the construction that $d'\Theta = \Theta d$ and that Θ induces $\{\theta^n\}$.

PROPOSITION 3. *If C' is an A-free chain group, $\{G^n | n \geqq 0\}$ is a collection of A modules and $H_n(C')$ is isomorphic to G^n for all $n \geqq 0$ then $H(T(C'))$ is ring isomorphic to $T^*(\sum G_n)$.*

PROOF. Let θ^n: $G^n \to H_n(C')$ be the isomorphism. We construct C as in Proposition 1, and Θ: $C \to C'$ as in Proposition 2. Then $\overline{\Theta}$: $T(C) \to T(C')$ given by $(\overline{\Theta} | C^{(n)} = \Theta \otimes \cdots \otimes \Theta$ is a ring homomorphism commuting with the boundary homomorphisms. By the Kunneth theorem, $\overline{\Theta}_*$: $H(C^{(p)}) \approx H(C'^{(p)})$ and hence $\overline{\Theta}_*$: $T^*(\sum G_n) \approx H(T(C'))$.

2. **The spaces X and Ω: notation and preliminary remarks.** X is a 1-connected topological space of Lusternik-Schnirelmann strong

category two. That is, there are closed subsets X_1, X_2 and maps $R_1, R_2\colon X\times I\to X$ such that

$$\begin{aligned} X &= X_1 \cup X_2, \\ R_i(x, 0) &= x \qquad\qquad\qquad (\text{for } x \in X,\ i = 1, 2), \\ R_i(X_i \times I) &\subset X_i, \qquad R_i(X \times 1) = x_0 \in X_1 \cap X_2. \end{aligned}$$

Since X is 1-connected, it imposes no additional restriction to assume $R_i(x_0\times I)=x_0$. We shall, however, impose the restriction that $(X;X_1,X_2)$ is to be a proper triad. In addition let

$$\begin{aligned} X_0 &= X_1 \cap X_2, \\ E &= (X, x_0)^{(I,1)};\ p\colon E \to X \text{ is the fibre map given by } p(f) = f(0), \\ \Omega &= p^{-1}(x_0), \\ E_i &= p^{-1}(X_i) \qquad\qquad (i = 0, 1, 2), \end{aligned}$$

$\rho_i\colon X_i \to E_i$ by $\rho_i(x)(t) = R_i(x, t)$ for $x \in X_i$, $t \in I$ $(i = 1, 2)$,

$\rho_0\colon X_0 \to \Omega$ by $\rho_0(x)\nu\rho_1(x) * \rho_2(x)$ for $x \in X_0$ (ν is defined below).

If f, g are paths in X with $f(1)=g(0)$ then $f*g$ is the path

$$f*g(t) = \begin{cases} f(2t), & 0 \leqq t \leqq 1/2, \\ g(2t-1), & 1/2 \leqq t \leqq 1. \end{cases}$$

If f is a path in X then νf is the path given by $\nu f(t)=f(1-t)$.

The homology theory used here is the cubical theory, with coefficients in the principal ideal ring A, described below:

An *n-cube* of a space X is a map $u\colon I^n\to X$. We say a cube is *degenerate* if there is an integer i, $1\leqq i\leqq n$, such that $u(t_1, \cdots, t_{i-1}, t_i, t_{i+1}, \cdots, t_n)=u(t, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_n)$ for $0\leqq t_i\leqq 1$. $Q_n(X)$ is the free A-module generated by n-cubes; $D_n(X)$ the subgroup of $Q_n(X)$ generated by degenerate cubes and $C_n(X)=Q_n(X)/D_n(X)$ is the group of cubical n-chains of X. If $1\leqq i\leqq n$ and $0\leqq\epsilon\leqq 1$ then $\lambda_{i,\epsilon}u$ is the $(n-1)$-cube, $\lambda_{i,\epsilon}u(t_1, \cdots, t_{n-1})=u(t_1, \cdots, t_{i-1}, \epsilon, t_i, \cdots, t_{n-1})$. The boundary on $C_n(X)$ is induced by

$$\partial u = \sum_{i=1}^{n} (-1)^i(\lambda_{i,1}u - \lambda_{i,0}u).$$

The homology groups given by this theory are isomorphic to the corresponding singular homology groups [1].

If u: $I^p\to\Omega$ and v: $I^q\to\Omega$ then $u*v$: $I^{p+q}\to\Omega$ is defined by $u*v(t_1, \cdots, t_{p+q})=u(t_1, \cdots, t_p)*v(t_{p+1}, \cdots, t_{p+q})$. This induces a pairing $C_p(\Omega)\otimes C_q(\Omega)\to C_{p+q}(\Omega)$, which yields the Pontrjagin product.

PROPOSITION 4. *Let* $F_i\colon E_i\to X_i\times\Omega$ *and* $G_i\colon X_i\times\Omega\to E_i$ $(i=1, 2)$ *be given by*

$$F_i(f) = (p(f), \nu\rho_i(p(f)) * f), \qquad f\in E_i,$$
$$G_i(x, f) = \rho_i(x) * f, \qquad x\in X_i, f\in\Omega.$$

Then (F_i, G_i) *are homotopy equivalences between* E_i *and* $X_i\times\Omega$. *Furthermore, when restricted they form homotopy equivalences between* E_0 *and* $X_0\times\Omega$.

PROOF.

$$F_i\circ G_i(x, f) = (x, \nu\rho_i(x) * (\rho_i(x) * f)),$$
$$G_i\circ F_i(f) = (\rho_i(p(f))*(\nu\rho_i(p(f))*f).$$

The homotopies of these maps to the identities are set up in the obvious manner.

Consider the following diagram:

$$\begin{array}{ccc}
H_{n-1}(\Omega; M) \xleftarrow{\partial_1} H_n(E, \Omega; M) & & H_n(X\times\Omega, x_0\times\Omega; M) \\
\downarrow i_{1*} & & \downarrow (i_4\times i)_* \\
H_n(E, E_1; M) & & H_n(X\times\Omega, X_1\times\Omega; M) \\
\uparrow i_{2*} & & \uparrow (i_3\times i)_* \\
Hn(E_2, E_0; M) & \xrightarrow{F'_{2*}} & H_n(X_2\times\Omega, X_0\times\Omega; M) \\
\downarrow \partial_2 & & \downarrow \partial_3 \\
H_{n-1}(E_0, \Omega; M) & \xrightarrow{F''_{2*}} & H_{n-1}(X_0\times\Omega, x_0\times\Omega; M)
\end{array}$$

M is an A-module.

∂_1 is the boundary homomorphism for the pair (E, Ω). It is an isomorphism for $n>0$ since E is contractible.

i_1: $(E, \Omega)\subset(E, E_1)$. Since X_1 is contractible, we have $H_n(E_1, \Omega)=0$ for $n\geqq 0$ and i_{1*} is an isomorphism (let $r_1(f)(t)=R_1(f(t), 1)$; then $r_{1*}=i_{1*}^{-1}$).

i_2: $(E_2, E_0)\subset(E, E_1)$ induces the isomorphism i_{2*} by excision.

∂_2 is the boundary homomorphism for the triple (E_2, E_0, Ω). Since X_2 is contractible we have $H_n(E_2, \Omega)=0$ for $n\geqq 0$ and ∂_2 is an isomorphism.

F'_{2*}, F''_{2*} are induced by F_2. Using Proposition 4 and the five Lemma, it is easy to see that these are isomorphisms.

i: $\Omega\subset\Omega$.

i_4: $(X, x_0) \subset (X, X_1)$. Note that $(i_4 \times i)_*$ is an isomorphism.

i_3: $(X_2, X_0) \subset (X, X_1)$; $(i_3 \times i)_*$ is an isomorphism by excision.

∂_3 is the boundary homomorphism of the triple $(X_2 \times \Omega, X_0 \times \Omega, x_0 \times \Omega)$; $\partial_3 = F''_{2*} \circ \partial_2 \circ (F'_{2*})^{-1}$ is an isomorphism.

We now have $(i_4 \times i)_*^{-1} \circ (i_3 \times i)_* \circ F'_{2*} \circ i_{1*} \circ \partial_1^{-1}$ is an isomorphism from $H(\Omega; M)$ onto $H_n((X, x_0) \times \Omega; M)$ and $\gamma = \partial_1 \circ (i_{1*})^{-1} \circ i_{2*} \circ G'_{2*} \circ \partial_3^{-1}$ is an isomorphism from $H_n((X_0, x_0) \times \Omega; M)$ onto $H_n(\Omega; M)$ for $n > 0$. This is a generalization of a lemma by G. W. Whitehead [**3**, p. 212].

3. **The main theorem.** We set $C = \bigtimes_{n=0}^{\infty} C_n(X_0, x_0)$ and suppose that $M = A$. To make the arguments that follow less cumbersome we replace the chain groups of the product spaces discussed in §2 by the tensor products of the chain groups of these spaces. The symbols formerly used to designate continuous functions should now be read as the corresponding chain maps.

Lemma 5. *Let* $\bar{\eta}$: $C \otimes C(\Omega) \to C(\Omega)$ *be given by* $\bar{\eta}(u \otimes v) = \rho_0 u * v$. *Then* $\bar{\eta}$ *induces the isomorphism* γ.

Proof. Let u be a p-cube of (X_0, x_0) and v a q-cube of Ω. We define Φ_1: $C_p(X_0, x_0) \otimes C_q(\Omega) \to C_{p+1}(X_2, X_0) \otimes C_q(\Omega)$ as follows. Let $u'(t_1, \cdots, t_{p+1}) = -R_2(u(t_2, \cdots, t_{p+1}), t_1) \in Q_{p+1}(X_2, X_0)$. Then Φ_1 is given by $\Phi_1(u \otimes v) = u' \otimes v$. Note that $\partial \Phi_1(u \otimes v) = u \otimes v - \Phi_1 \partial(u \otimes v)$ modulo degenerate cubes and so Φ_1 induces ∂_3^{-1}. Thus $(\partial_1 \circ r_1 \circ i_2 \circ G_2 \circ \Phi_1)_* = \gamma$. Next,

$$\begin{aligned}\partial(r_1 \circ i_2 \circ G_2 \circ \Phi_1(u \otimes v)) &= r_1 \circ i_2 \circ G_2(\partial \Phi_1(u \otimes v)) \\ &= r_1 \circ i_2 \circ G_2(u \otimes v) - r_1 \circ i_2 \circ G_2 \Phi_1 \partial)(u \otimes v) \\ &= r_1(\rho_2 u * v) - r_1 \circ i_2 \circ G_2 \circ \Phi_1 \partial(u \otimes v)\end{aligned}$$

which is a chain in $C_{p+q}(E)$. Let Φ_2: $C_{n-1}(E) \to C_n(E)$ be given by

$$\Phi_2 w(t, \cdots, t_n)(t) = R_1(w(t_2, \cdots, t_n)(h_1(t_1, t)), h_2(t_1, t))$$

when w: $I^{n-1} \to E$ and h_1, h_2: $I^2 \to I$ are maps such that

$$h_1(s, t) = \begin{cases} t & \text{for } s = 0, \quad 0 \leqq t \leqq 1, \\ 0 & \phantom{\text{for }} s = 1, \quad 0 \leqq t \leqq 1/2, \\ 2t - 1 & \phantom{\text{for }} s = 1, 1/2 \leqq t \leqq 1, \end{cases}$$

$$h(s, t) = \begin{cases} 1 & s = 0, \quad 0 \leqq t \leqq 1, \\ 1 - 2t & s = 1, \quad 0 \leqq t \leqq 1/2, \\ 0 & s = 1, 1/2 \leqq t \leqq 1. \end{cases}$$

Thus we have $\partial\Phi_2 w+\Phi_2\partial w=(\nu\rho_1 p w)*w-r_1 w$. In particular, setting $w=\rho_2 u*v$, $n=p+q+1$ we have

$$(\partial\Phi_2+\Phi_2\partial)(\rho_2 u*v)=(\nu\rho_1 u)*(\rho_2 u*v)-r_1(\rho_2 u*v).$$

Finally there is a homotopy of Ω into itself sending $(\nu\rho_1(x))*(\rho_3(x)*f)$ into $\rho_0(x)*f$ when $x\in X_0$, $f\in\Omega$. Combining these homotopies yields: there is a homomorphism

$$\Phi: C_p(X_0, x_0)\otimes C_q(\Omega)\rightarrow C_{p+q+1}(\Omega)$$

such that

$$(\partial\Phi+\Phi\partial)(u\otimes v)=(\partial\circ r_1\circ i_2\circ G_2\circ\Phi_1)(u\otimes v)-(\rho_0 u)*v.$$

Hence the homomorphism $\bar{\eta}$ induces γ.

Let η_n: $C^{(n)}\rightarrow C(\Omega)$ be defined by $\eta_0(a)=a\omega_0$ (where ω_0 is a basic 0-cocycle of Ω and $a\in A=C^{(0)}$), $\eta_n(c_1\otimes\cdots\otimes c_n)=\rho_0 c_1*\cdots*\rho_0 c_n$ for $n>0$. Extend $\{\eta_n\}$ to η: $T(C)\rightarrow C(\Omega)$; this is a ring homomorphism commuting with the boundary operators and hence induces η_*: $T^*(H(X_0, x_0))\rightarrow H(\Omega)$.

Proposition 6. *η_* is an isomorphism.*

Proof. Note that $T(C)$ is graded by dim $a=0$ if $a\in C^{(0)}$, dim $(c_{n_1}\otimes\cdots\otimes c_{n_k})=n_1+\cdots+n_k$ when $c_{n_i}\in C_{n_i}(X_0, x_0)$ $(i=1,\cdots,k)$, and η preserves this grading. The proof now goes by induction on this dimension.

Clearly η_*: $H_0(T(C))\approx H_0(\Omega)$. Next suppose η_*: $H_{n-1}(T(C))\approx H_{n-1}(\Omega)$. Since $H_0(C^{(1)})=0$, it follows that $H_{n-1}(T(C))=H_{n-1}(\sum_{p<n}C^{(p)})$. Thus for $n\geqq 1$, $H_n(T(C))\approx H_n(C^{(1)}\otimes\sum_{p<n}C^{(p)})$. Using the induction assumption,

$$(1\otimes\eta_{n-1})_*: H_n\left(C^{(1)}\otimes\sum_{p<n}C^{(p)}\right)\approx H_n(C\otimes C(\Omega)).$$

From Lemma 5,

$$\bar{\eta}_*: H_n(C\otimes C(\Omega))\approx H_n(\Omega).$$

But $\bar{\eta}\circ(1\otimes\eta_{n-1})=\eta_n$ and so we have η_*: $H_n(T(C))\approx H_n(\Omega)$.

Since $H_n(X)\approx H_{n-1}(X_0, x_0)$ for $n\geqq 1$ we may reword this proposition to read:

Theorem 7. *If X is a 1-connected space of Lusternik-Schnirelmann strong category two and Ω the loop space on X then the Pontrjagin ring $H_*(\Omega)$ is isomorphic to $T^*(\sum_{n=2}^{\infty}H_n(X))$.*

In case G is torsion free, the torsion products in $T^*(G)$ vanish so that $T^*(G)=T(G)$. Thus if $H_*(X)$ is torsion free we have $H_*(\Omega) \approx T(H_*(X, x_0))$. This result was known to Bott and Samelson [1].

BIBLIOGRAPHY

1. H. Bott and H. Samelson, *On the Pontrjagin product in spaces of paths*, Comment. Math. Helv. vol. 27 (1953) pp. 320–337.

2. N. Bourbaki, *Algèbre multilinéaire*, Actualités Scientifiques et Industrielles, vol. 1044.

3. G. W. Whitehead, *On the Freudenthal theorems*, Ann. of Math. vol. 57 (1953) pp. 209–228.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY AND
PURDUE UNIVERSITY