A GENERALIZATION OF THE WEDDERBURN-MALCEV THEOREM TO INFINITE DIMENSIONAL ALGEBRAS[1]

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1. **Summary.** In this paper we extend the Wedderburn Principal Theorem and the Malcev Theorem for associative algebras to certain infinite dimensional algebras. Let A be an algebra over the base field F having (Jacobson) radical N. The Wedderburn Principal Theorem states that if N is nilpotent and if A/N is a finite dimensional separable algebra over F, then A is cleft. The Malcev Theorem, as generalized by Tihomirov [7], states that under these hypotheses any two cleavings of A are conjugate. (See §2 for the definitions of these terms.) Curtis [3] has generalized these theorems to the case where $\bigcap_{k=1}^{\infty} N^k=0$ and A is complete with respect to the topology in which the powers of N form a fundamental system of neighborhoods of zero. In §3 we show that the Wedderburn Principal Theorem holds for such an algebra A in the case where A/N has countable dimension over the base field. To generalize the Malcev Theorem, we drop this dimensionality restriction, but the following hypothesis on the radical is needed: for every positive integer n, N^n/N^{n+1} is assumed to be complete with respect to a topology in which a fundamental system of neighborhoods of zero is composed of centralizers of finite dimensional separable subalgebras of A/N (when N^n/N^{n+1} is considered as an A/N-module). With these conditions it is shown in §4 that any two cleavings of A are conjugate. The necessity for this additional hypothesis in the Malcev Theorem is shown in §5 by an example in which A/N is of dimension $\aleph_0$ over F and $N^2=0$.

2. **Introduction.** By an algebra A over the base field F we shall mean an associative algebra of possibly infinite dimension over F. The (Jacobson) radical of A shall be denoted by N. To denote the direct sum of two vector subspaces B and C of A, when A is considered as a vector space over F, we shall use the notation $B\oplus C$. The algebra A will be called *locally separable* in case every finite set of elements of A is contained in a finite dimensional (over F) separable subalgebra of A.

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If $r\in A$ has a quasi-inverse r', then the mapping $a\to a^r=a-ar-r'a+r'ar$, for $a\in A$, is an automorphism of A, called the *quasi-inner automorphism of A generated by r*. (Note that $a^r=(1-r)^{-1}a(1-r)$ if A has an identity element.) If $r\circ s=r+s-rs$, then it is easy to verify that if r and s have quasi-inverses, $(a^r)^s=a^{r\circ s}$, for $a\in A$. If the algebra A contains a subalgebra S such that $A=S\oplus N$, then A is said to be *cleft* and such an expression for A is called a *cleaving* of A. Two cleavings of A, say $A=S\oplus N=S^*\oplus N$, are said to be *conjugate* if S can be mapped onto S^* by a quasi-inner automorphism of A generated by an element of N.

A topology can be imposed on A by taking the sets N^k $(k=1, 2, \cdots)$ as a fundamental system of neighborhoods of zero. If we assume that the radical N has the property that $\bigcap_{k=1}^{\infty} N^k=0$, then A with respect to this topology is a Hausdorff topological ring. In this case we shall call the topology the *N-adic topology* of A. With the N-adic topology A is a metrizable space, so the usual idea of completeness in terms of Cauchy sequences is applicable.

3. A generalization of the Wedderburn Principal Theorem.

LEMMA. *Let A be an algebra over the base field F having radical N with $\bigcap_{k=1}^{\infty} N^k=0$, such that A is complete with respect to the N-adic topology and A/N is a finite dimensional separable algebra over F. Let E be a closed subalgebra of A, such that $E/(E\cap N)$ is a finite dimensional separable algebra over F. Then for a cleaving $E=D\oplus(E\cap N)$ of E, there exists a cleaving of A, $A=B\oplus N$, such that $B\supset D$.*

PROOF. Since E is closed, $E\cap N$ is a (right) quasi-regular ideal of E. Also $E/(E\cap N)$ is semi-simple, so $E\cap N$ is the radical of E. Let $H=D\oplus N\subset A=B'\oplus N$, where $B'\oplus N$ is a cleaving of A (which exists by the result of Curtis [3] mentioned in §1). This gives another cleaving of H, $H=D'\oplus N$ with $D'\subset B'$. $D\cong E/(E\cap N)$ which is a finite dimensional separable algebra over F. Since N is open, $H=D\oplus N$ is an open subalgebra, hence it is closed. Thus, H is complete with respect to the induced topology. By Curtis' results there exists a quasi-inner automorphism of H generated by an element of N that maps D' onto D. This can be extended to A and it will map B' onto some B such that $D\subset B$. This proves the lemma.

THEOREM 1. *Let A be an algebra over the base field F having radical N with $\bigcap_{k=1}^{\infty} N^k=0$, such that A is complete with respect to the N-adic topology and such that A/N is a locally separable algebra of dimension $\leq\aleph_0$ over F. Then A is cleft.*

PROOF. By the hypotheses there exists in A/N a nondecreasing sequence of finite dimensional separable subalgebras Γ_k over $F(k=1, 2, \cdots)$ such that $A/N=\bigcup_{k=1}^{\infty} \Gamma_k$. Let $\bar{a}_1, \bar{a}_2, \bar{a}_3, \cdots$ be a basis of A/N over F such that the first k_n elements form a basis of Γ_n over F $(n=1, 2, \cdots)$. In A take representatives $a_1, a_2, a_3, \cdots$ of these and for each n let C_n be the closure in A for the N-adic topology of the subalgebra generated over F by $a_1, a_2, \cdots, a_{k_n}$. It is easily seen that under the canonical mapping of A onto A/N, C_n maps onto Γ_n. Also $\bigcap_{k=1}^{\infty} (C_n\cap N)^k=0$, and since C_n is closed, $C_n\cap N$ is the radical of C_n. Thus, C_n is complete with respect to the induced topology and by Curtis' result there exists a subalgebra D_n of C_n such that $C_n=D_n\oplus(C_n\cap N)$.

Suppose we have the cleavings $C_i=D_i\oplus(C_i\cap N)$, $i=1, 2, \cdots, n-1$, such that $D_1\subset D_2\subset \cdots \subset D_{n-1}$. Consider $C_n\supset C_{n-1}$. By the lemma and the above results, there exists a cleaving $C_n=D_n\oplus(C_n\cap N)$ such that $D_{n-1}\subset D_n$. Hence, we have a nondecreasing sequence of subalgebras of A, $D_1\subset D_2\subset D_3\subset \cdots$, such that $C_n=D_n\oplus(C_n\cap N)$, $n=1, 2, \cdots$. Take $D=\bigcup_{k=1}^{\infty} D_k$. Then $A=D\oplus N$ is the desired cleaving.

4. **A generalization of the Malcev-Tihomirov Theorem.** Let A be an algebra over the field F having radical N. If M is any two-sided A/N-module, we define for $\bar{a}\in A/N$

$$M(\bar{a}) = \{m \in M \mid m\bar{a} = \bar{a}m\},$$

and for $\Gamma\subset A/N$,

$$M(\Gamma) = \bigcap_{\bar{a}\in\Gamma} M(\bar{a}).$$

Note that $M(\bar{a})$ and $N(\Gamma)$ are additive subgroups of M.

THEOREM 2. *Let A be an algebra over the base field F having radical N with $\bigcap_{k=1}^{\infty} N^k=0$, such that A is complete with respect to the N-adic topology and such that A/N is locally separable over F. For every positive integer n, $N_n=N^n/N^{n+1}$ is assumed to be complete with respect to the topology $\mathfrak{J}_n$ in which the sets $N_n(\Gamma)$, for Γ running through all finite dimensional separable subalgebras of A/N, form a fundamental system of neighborhoods of zero. Then any two cleavings of A are conjugate.*

REMARK. It is easily seen that the sets $N_n(\Gamma)$, for Γ running through all finite dimensional separable subalgebras of A/N, form a fundamental system of neighborhoods of zero for a topology on N_n. However, it is not, in general, a Hausdorff topology. We take completeness in the sense of Bourbaki [2, Chap. 2, §3] and we shall use the Bourbaki terminology throughout the proof.

PROOF. Suppose the cleavings of A are $A=S\oplus N=S^*\oplus N$. If $a\in S$, by a^* we mean that element of S^* such that $a\equiv a^*$ (mod N).

CASE 1. Assume that $N^2=0$. In this case the quasi-inner automorphism of A generated by $r\in N$ takes the form $a^r=a-ar+ra$, $a\in A$. Considering N as an A/N-module, we have that two elements of N generate quasi-inner automorphisms of A mapping $a\in A$ onto the same element if and only if their difference lies in the N-module $N(\bar{a})$.

Let C be a finite dimensional separable subalgebra of S with $\Gamma\subset A/N$ its image under the canonical mapping of A onto A/N. Then there exists a finite dimensional separable subalgebra C^* of S^*, such that $C\oplus N=C^*\oplus N$. By the Malcev-Tihomirov Theorem [7], there exists a quasi-inner automorphism of $C\oplus N$ generated by an element $r\in N$ such that $C^r=C^*$. The set of all $r\in N$ generating this quasi-inner automorphism we shall call R_Γ. By the above results R_Γ is just a coset of $N(\Gamma)$ in N.

We next prove that the collection $\{R_\Gamma\}$, where Γ runs through all finite dimensional separable subalgebras of A/N, forms a filter base. Let Γ and Γ' be two finite dimensional separable subalgebras of A/N. By the local separability of A/N, there exists a finite dimensional separable subalgebra Δ of A/N containing Γ and Γ'. Then clearly $R_\Delta\subset R_\Gamma\cap R_{\Gamma'}$, so $\{R_\Gamma\}$ is a filter base. When we consider N with the topology $\mathfrak{I}$ formed by taking the sets $N(\Gamma)$ as neighborhoods of zero, we see that $\{R_\Gamma\}$ forms a Cauchy filter base because R_Γ is a coset of $N(\Gamma)$. By the completeness of N, this filter base has a limit $r\in N$. Therefore, r is in the closure of each R_Γ; and since R_Γ, being an open coset, is closed, $r\in\bigcap R_\Gamma$. Then $S^r\subset S^*$. Since the mapping is clearly onto, we have $S^r=S^*$, which concludes the proof of Case 1.

REMARK. The completeness hypothesis on N is not unduly strong because convergence of *all* Cauchy filters is equivalent to convergence of Cauchy filters with a base of cosets of the $N(\Gamma)$'s. Indeed, let $\mathfrak{F}$ be any Cauchy filter in N for the topology $\mathfrak{I}$. For each $N(\Gamma)$, $\mathfrak{F}$ contains exactly one coset of $N(\Gamma)$; let $\mathfrak{F}'$ be the collection of these cosets. Then $\mathfrak{F}'$ is a Cauchy filter base, and being coarser then $\mathfrak{F}$, its convergence implies that of $\mathfrak{F}$. On the other hand, if $\mathfrak{F}$ converges to $r\in N$, r is a limit point of $\mathfrak{F}$, hence of $\mathfrak{F}'$, so it is a limit of $\mathfrak{F}'$. Thus, $\mathfrak{F}$ converges if and only if $\mathfrak{F}'$ converges.

CASE 2. We now remove the restriction that $N^2=0$. We use induction. Suppose there exist elements $r_i\in N^i$, $i=1, 2, \cdots, n-1$, such that

$$a^{r_1\circ\cdots\circ r_{n-1}}\equiv a^* \pmod{N^n}, \qquad \text{for all } a\in S. \tag{1}$$

Let S' be the image of S under the quasi-inner automorphism of A generated by $r_1 \circ \cdots \circ r_{n-1}$. Then $S' \oplus N^n = S^* \oplus N^n$. Let ψ_n be the canonical mapping of A onto A/N^{n+1}. (We shall write the mapping symbol on the right.) We define $N_n = N^n\psi_n = N^n/N^{n+1}$ and $A_n = S'\psi_n \oplus N_n = S^*\psi_n \oplus N_n$. Then $N_n^2 = 0$ and A_n/N_n is semi-simple and locally separable. Therefore, N_n is the radical of A_n and we can apply the process of Case 1 to the algebra A_n. We thus have a collection $\{R_{n,\Gamma}\}$ of cosets of $N_n(\Gamma)$ in N_n, where Γ runs through all finite dimensional separable subalgebras of A/N. This collection is a Cauchy filter base for the topology $\mathfrak{J}_n$. By the hypothesis of completion of N_n with respect to this topology, there exists an element $r_n \in N^n$ such that $r_n\psi_n \in \bigcap R_{n,\Gamma}$. Then $r_n\psi_n$ generates a quasi-inner automorphism of A_n that maps $S'\psi_n$ onto $S^*\psi_n$, that is, for this $r_n \in N^n$ the relation (1) is true with n replaced by $n+1$. This completes the induction step.

We have thus constructed a sequence $\{r_n\}$, $r_n \in N^n$ $(n=1, 2, \cdots)$ such that (1) holds for each positive integer n. Hence, with respect to the N-adic topology, the sequence $\{a^{r_1 \circ \cdots \circ r_n} - a^*\}$ has limit 0. The sequence r_1, $r_1 \circ r_2$, $r_1 \circ r_2 \circ r_3, \cdots$ is a Cauchy sequence in A for the N-adic topology and hence has a limit $r \in N$. Also the sequence obtained from this one by taking quasi-inverses of each term is a Cauchy sequence and converges to $r' \in N$. For each $a \in S$ we have $a^* - a^{r_1 \circ \cdots \circ r_n} = a^* - a + a(r_1 \circ \cdots \circ r_n) + (r_1 \circ \cdots \circ r_n)'a - (r_1 \circ \cdots \circ r_n)'a(r_1 \circ \cdots \circ r_n)$. Taking limits of both sides with respect to the N-adic topology of A, we get $a^* = a^r$. Since this is true for each $a \in S$, we have $S^r \subset S^*$ and clearly this mapping is onto. Thus, $r \in N$ generates a quasi-inner automorphism of A which maps S onto S^*. This proves Theorem 2.

COROLLARY 1. *Let A be an algebra over the base field F having radical N with $\bigcap_{k=1}^{\infty} N^k = 0$, such that A is complete with respect to the N-adic topology and such that A/N is a finite dimensional separable algebra over F. Then any two cleavings of A are conjugate.*

PROOF. For any positive integer n, all the $N_n(\Gamma)$ can be deleted except $N_n(A/N)$ itself and the topology $\mathfrak{J}_n$ is unchanged. A Cauchy filter on N_n for the topology $\mathfrak{J}_n$ contains a coset of $N_n(A/N)$ and any element of this coset is a limit of the filter. Hence, N_n is complete and the result follows from Theorem 2.

Corollary 1 is Curtis' generalization of the Malcev Theorem [3]. Corollary 2, which follows below, is a result of Kuročkin [5].

COROLLARY 2. *Let A be an algebra over the base field F having a finite dimensional radical N such that A/N is locally separable. Then any two cleavings of A are conjugate.*

PROOF. Since N is finite dimensional, it is nilpotent, i.e., there exists a positive integer m such that $N^m=0$. From this it easily follows that A is complete with respect to the N-adic topology. Let n be a positive integer smaller than m. If Γ and Γ' are finite dimensional separable subalgebras of A/N such that $\Gamma \supset \Gamma'$, then $N_n(\Gamma) \subset N_n(\Gamma')$. Since N_n is finite dimensional and A/N is locally separable, there exists a finite dimensional separable subalgebra Δ of A/N such that $N_n(\Delta)$ has minimal dimension in the set of all $N_n(\Gamma)$. Although Δ is not unique, it is easily seen that all such minimal $N_n(\Delta)$ are equal. Thus a fundamental system of neighborhoods of zero consists just of the set $N_n(\Delta)$. As in the proof of Corollary 1, it follows that N_n is complete with respect to the topology $\mathfrak{I}_n$. The conclusion of the corollary now follows from the theorem.

5. **Example.** We shall construct an algebra A over the field F of rational numbers having radical N with $N^2=0$, such that A/N is a separable field of dimension $\aleph_0$ over F, but such that A has two non-conjugate cleavings. This will show the necessity of having some additional hypothesis on the radical, as in Theorem 2, in order that the Malcev Theorem be true when A/N is not finite dimensional. Note that since $N^2=0$, A will be complete with respect to the N-adic topology.

Let ζ_n be a primitive 2^{n+1}th root of unity for $n=0, 1, 2, \cdots$, and $\zeta_n=1$ for $n<0$. For each nonnegative integer n, we consider the field $K_n=F(\zeta_n)$ and let $K=\bigcup_{n=0}^{\infty} K_n$. K is then a separable field of dimension $\aleph_0$ over F. Let N be the set of all $\aleph_0$-tuples over K in which there are only a finite number of nonzero terms. With the usual operations N is a vector space over K; let e_n be the $\aleph_0$-tuple with 1 in the nth place and zeros elsewhere. We define A to be the direct sum of the F-spaces K and N. To make A into an algebra, we define multiplication as follows:

(1) For elements in K, multiplication is the field multiplication.

(2) For elements in N, we define $N^2=0$.

(3) For $k\in K$ and $r=\sum k_n e_n \in N$ (where the k_n are in K and all but a finite number are zero), we define $kr=\sum (kk_n)e_n$.

(4) For each $j=1, 2, \cdots$, let σ_j be the F-automorphism of the field K such that $\sigma_j(\zeta_n)=\zeta_n^{2^j+1}$, for each n. Then for every $k\in K$ we define $e_j k=\sigma_j(k)e_j$.

(5) Lastly if $r=\sum k_j e_j \in N$, we define $rk=\sum k_j \sigma_j(k) e_j$.

It can be shown that A is then an algebra over F having an identity element, namely the identity element of K, and that N is the radical of A. Also A/N is isomorphic to K and $A=K\oplus N$ is a cleaving of A.

We shall now give another cleaving of A. For each non-negative integer n, we define $r_n = \sum_j(\zeta_n e_j - e_j\zeta_n)$, which is an element of N. By an obvious induction on q, we find that $(\zeta_n - r_n)^{2^q} = \zeta_n^{2^q} - \sum_j(\zeta_n^{2^q} e_j - e_j\zeta_n^{2^q})$, from which it follows that $\zeta_n - r_n$ is a primitive 2^{n+1}th root of unity. For each n we consider the field $K_n^* = F(\zeta_1 - r_1, \zeta_2 - r_2, \cdots, \zeta_n - r_n)$ and let $K^* = \bigcup_{n=0}^{\infty} K_n^*$, where $K_0^* = F$. Then for each n, $K_n \oplus N = K_n^* \oplus N$, and $A = K^* \oplus N$.

Now let $s \in N$ and consider $(1+s)\zeta_n(1-s)$. This is a (primitive) 2^{n+1}th root of unity. Hence if it lies in K^*, it must be a power, say the mth, of $\zeta_n - r_n$, that is, $(\zeta_n - r_n)^m = \zeta_n - (\zeta_n s - s\zeta_n)$. Taking cosets mod N, we have that $\zeta_n^m = \zeta_n$, hence also $(\zeta_n - r_n)^m = \zeta_n - r_n$. Thus we have $\zeta_n s - s\zeta_n = r_n$. If we write $s = \sum k_j e_j$, with $k_j \in K$, we have from this that $\zeta_n k_j - k_j \sigma_j(\zeta_n) = \zeta_n - \sigma_j(\zeta_n)$, $j = 1, 2, \cdots$, that is, $\zeta_n k_j(1 - \zeta_n^{2^j}) = \zeta_n(1 - \zeta_n^{2^j})$. If $j \leqq n$, $1 - \zeta_n^{2^j} \neq 0$, so $k_j = 1$. However, since N consists of the $\aleph_0$-tuples with only a finite number of nonzero terms, this shows that if $s \in N$, the isomorphism $k \rightarrow (1+s)k(1-s)$ cannot map every ζ_n into K^*. Hence, $A = K \oplus N = K^* \oplus N$ are two nonconjugate cleavings of A.

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