ON A PROPERTY OF MONOTONE AND CONVEX FUNCTIONS

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We shall deal with real functions which have continuous second derivatives in some open interval (a, b), $-\infty \leqq a < b \leqq \infty$. The interval of definition of $f(x)$ is denoted by $I(f)$. $\phi(x)$ is called convex (from below) if $\phi''(x) \geqq 0$, concave if $\phi''(x) \leqq 0$ in $I(\phi)$.

If $\phi(x)$ is monotone increasing and convex, and $\psi_1(x)$ is monotone increasing and concave such that the range of $\phi(x)$ is contained in $I(\psi_1)$, then

$$f(x) = \psi_1(\phi(x)) \tag{1}$$

is also monotone increasing, but usually neither convex nor concave. The question arises, under what conditions can $f(x)$ be represented in the form (1).

THEOREM 1. *If $f(x)$ is strictly monotone increasing and has a continuous second derivative in $I(f)$ then it has a representation* (1).

Theorem 1 states that there is a strictly increasing concave function $\psi(u)$ with continuous second derivative such that $\psi_1(u) = f(\psi(u))$ is concave. This is equivalent to

$$\psi_1''(u) = f''(\psi(u))[\psi'(u)]^2 + f'(\psi(u))\psi''(u) \leqq 0, \tag{2}$$

or if $\phi(x)$ is the inverse of $\psi(u)$, to

$$f''(x)/f'(x) \leqq \phi''(x)/\phi'(x). \tag{3}$$

Let $f_+''(x)$ denote $f''(x)$ if $f''(x) \geqq 0$ and 0 if $f''(x) < 0$. Consider the function

$$\phi_0(x) = \int_d^x e^{p(y)}dy \tag{4}$$

where d is any fixed number in (a, b) and

$$p(y) = \int_d^y [f_+''(t)/f'(t)]dt. \tag{5}$$

Clearly $\phi_0'(x) > 0$ and

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(6) $$\phi_0''(x)/\phi_0'(x) = f_+''(x)/f'(x) \geqq f''(x)/f'(x)$$

so that (3) is satisfied, also

(7) $$\phi_0''(x) \geqq 0.$$

This proves the theorem.[1]

If $f(x)$ is bounded and $I(f)$ is finite, the question comes up whether $\phi(x)$ itself can be chosen to be bounded. This is answered by

THEOREM 2. *If $f(x)$ is bounded, strictly increasing and has a continuous second derivative in (a, b), then it can be represented in the form* (1) *with bounded $\phi(x)$ if and only if the integral*

(8) $$\int_d^b e^{p(y)}dy$$

converges, where $a<d<b$ and $p(y)$ is the function defined under (5).

We shall see presently that boundedness of $f(x)$ does not necessarily imply finiteness of (8).

To prove Theorem 2 we first note that, by (4), $\phi_0(x)$ is bounded from above if (8) is finite, and also bounded from below if $f(x)$ is bounded since

$$\begin{aligned}\phi_0(x) &\geqq \int_d^x \left\{\exp \int_d^y [f''(t)/f'(t)]dt\right\} dy \\ &= [f(x) - f(d)]/f'(d)\end{aligned}$$

by (4) and (5).

Suppose now that (8) diverges, so that $\phi_0(x)$ is unbounded from above, and let $\phi_1(x)$ be any function which has the properties (3) and (7). By taking a suitable linear combination $\phi(x)=c_1\phi_1(x)+c_2$ we can achieve that $\phi(d)=0$, $\phi'(d)=1$. Now from (3) and (6),

$$\frac{d}{dx}\log \phi'(x) \geqq f''(x)/f'(x) \geqq \frac{d}{dx}\log \phi_0'(x)$$

which implies

$$\phi'(x) \geqq \phi_0'(x), \qquad \phi(x) \geqq \phi_0(x) \qquad \text{for } x > d.$$

This shows that $\phi_0(x)$ is in a sense the "least convex" among all possible solutions and that $\phi(x)$, hence also $\phi_1(x)$, is unbounded.

[1] I am indebted to G. Lorentz for a substantial shortening of the original argument. His proof, which is reproduced above, contributed greatly to a simplified treatment of another part of the paper.

Theorems 1 and 2 have obvious dual formulations.

THEOREM 1.* *Under the same conditions as in Theorem* 1, $f(x)$ *can be represented in the form*

$$f(x) = \phi_1(\psi(x)) \tag{1*}$$

where ϕ_1 *is convex and* ψ *concave.*

THEOREM 2*. *If* $f(x)$ *is as in Theorem* 2, *then it can be represented in the form* (1*) *with bounded* $\psi(x)$ *if and only if*

$$\int_a^d e^{q(y)}\,dy \tag{8*}$$

is finite, where

$$q(y) = \int_y^d [f''_-(t)/f'(t)]dt. \tag{5*}$$

Here $f''_-(t)$ *denotes* $-f''(t)$ *if* $f''(t) \leqq 0$ *and* 0 *if* $f''(t) > 0$.

The following example shows that boundedness of $f(x)$ does not necessarily imply finiteness of (8) or (8*). Take $f(x) = 2x + x^2 \sin(1/x)$ over the interval (0, 1). It is easily seen that $f''(x)$ has a zero x_n between $2/(2n+1)\pi$ and $2/(2n-1)\pi$, and

$$f''(x)\begin{cases} >0 & \text{for} \quad x_{2m} < x < x_{2m-1}, \\ <0 & \text{for} \quad x_{2m+1} < x < x_{2m}, \qquad m = 1, 2, \cdots. \end{cases}$$

It can also be shown easily that

$$x_n = 1/n\pi + 2/n^3\pi^3 + O(n^{-5}),$$
$$f'(x_n) = 2 - (-1)^n + O(n^{-2}),$$

so that

$$\int_{x_{2m+1}}^{x_{2m}} [f''(t)/f'(t)]dt = -\log 3 + O(m^{-2})$$

and

$$q(y) = m \log 3 + O(1) \quad \text{for} \quad x_{2m+1} < x < x_{2m-1}.$$

This shows that $\int_0^{x_1} e^{q(y)}dy$ is divergent.

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