SOME TAUBERIAN PROPERTIES OF HÖLDER TRANSFORMATIONS

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1. **Introduction.** The result which follows was proved by me, recently, in [1],[2] Theorem (9.2). If, for some $\alpha > -1$, the sequence $\{s_n\}$, $n=0, 1, 2, \cdots$, is summable $A^{(\alpha)}$ to s, that is

$$\sum_{n=0}^{\infty} \binom{n+\alpha}{n} s_n z^n$$

is convergent in the unit circle and

$$(1.1) \qquad \lim_{x \uparrow 1} (1-x)^{\alpha+1} \sum_{n=0}^{\infty} \binom{n+\alpha}{n} s_n x^n = s \quad (|s| < +\infty),$$

and, for some pair β, γ of real numbers with $\beta < \gamma$,

$$(1.2) \qquad \lim_{n \to \infty} (h_n^{(\beta)} - h_n^{(\gamma)}) = 0$$

then

$$(1.3) \qquad \lim_{n \to \infty} h_n^{(\beta)} = s,$$

when generally[3] $\{h_n^{(\delta)}\}$ denotes the sequence of the Hölder transform of order δ of $\{s_n\}$.

The Hölder transform of order α $\{h_n^{(\alpha)}\}$ (or, in short, the (H, α) transform), where α is a real number, is defined as the Hausdorff transform generated by the sequence $\mu_n = (n+1)^{-\alpha}$, $n=0, 1, 2, \cdots$. It is known that the Hölder transformations are regular for $\alpha \geqq 0$. We say, too, that a sequence $\{s_n\}$ is summable Hölder to s if it is summable (H, α) to s for some real number α.

In this note we obtain an extension of my above result to the case

Received by the editors January 19, 1955 and, in revised form, June 23, 1955.

[1] This note is in part a part of a manuscript *On relations between methods of summability of infinite series and integrals* submitted as a thesis for the degree of Ph.D. to the Hebrew University, Jerusalem.

[2] Numbers in brackets refer to the bibliography at the end of this note.

[3] Concepts and propositions mentioned or used in this paper without definition or proof are to be found in Hardy's book *Divergent series*, Oxford University Press.

where $\{h_n^{(\gamma)}\}$ is replaced by certain linear combinations of Hölder transforms as well as some other results of similar type (see §§8 and 9).

In the proof of the extensions just mentioned we use some properties of general and special Hausdorff transformations as well as two tauberian propositions for $A^{(\alpha)}$ methods of summability, formulated below in §§2–7.

2. Some properties of the $J^{(\alpha,\beta)}$ and $J\left(\alpha;\begin{matrix}\beta,\gamma\\ a,b\end{matrix}\right)$ transformations.

For any pair α, β of real numbers, with $\alpha<\beta$, we call (see [1, p. 378]) the linear transform $\{t_n^{(\alpha,\beta)}\}$ of $\{s_n\}$,

$$t_0^{(\alpha,\beta)} = h_0^{(\alpha)}; \qquad t_n^{(\alpha,\beta)} = h_0^{(\alpha)} + \sum_{v=1}^{n} \frac{h_v^{(\alpha)} - h_v^{(\beta)}}{(\beta-\alpha)v}, \qquad n>0,$$

the $J^{(\alpha,\beta)}$ transform of $\{s_n\}$. $\{s_n\}$ is called summable $J^{(\alpha,\beta)}$ to s if $\lim_{n\to\infty} t_n^{(\alpha,\beta)}=s$.

The following two properties of $J^{(\alpha,\beta)}$ transforms were proved by me in [1, Theorem (9.1) and Theorem (9.2)].

THEOREM A. *For any pair α, β of real numbers, with $\alpha<\beta$, the $J^{(\alpha,\beta)}$ transform of $\{s_n\}$ is a Hausdorff transform generated by $\{\mu_n\}$, where $\mu_0=1$; $\mu_n=((n+1)^{-\alpha}-(n+1)^{-\beta})/(\beta-\alpha)n$, $n>0$.*

THEOREM B. *For any pair α, β of real numbers, with $\alpha<\beta$, the $J^{(\alpha,\beta)}$ method of summability is equivalent to the $(H,\alpha+1)$ method; thus, for $-1\leqq\alpha<\beta$, the $J^{(\alpha,\beta)}$ transformation is regular.*

A simple consequence of Theorem A and Theorem B is

THEOREM 2.1. *Let a, b, α, β and γ be five real numbers satisfying $a+b=1$, $\alpha<\beta<\gamma$, $0\leqq a$ and $a\alpha+b\beta>0$. Let $\{s_n\}$ be an arbitrary sequence. The linear transform*

$$\left\{ U_n^{\left(\alpha;\begin{smallmatrix}\beta,\gamma\\ a,b\end{smallmatrix}\right)} \right\}$$

defined by

$$U_0^{\left(\alpha;\begin{smallmatrix}\beta,\gamma\\ a,b\end{smallmatrix}\right)} = h_0^{(\alpha)}; \qquad U_n^{\left(\alpha;\begin{smallmatrix}\beta,\gamma\\ a,b\end{smallmatrix}\right)} = h_0^{(\alpha)} + \sum_{v=1}^{n} \frac{h_v^{(\alpha)} - [a h_v^{(\beta)} + b h_v^{(\gamma)}]}{[a(\beta-\alpha)+b(\gamma-\alpha)]v}, \quad n>0,$$

is a Hausdorff transform of $\{s_n\}$ generated by the sequence $\{\mu_n\}$, $\mu_0=1$; $\mu_n=\{[a(\beta-\alpha)+b(\gamma-\alpha)]n\}^{-1}\cdot[(n+1)^{-\alpha}-a(n+1)^{-\beta}-b(n+1)^{-\gamma}]$, $n>0$.

Definition. The linear Hausdorff transform

$$\left\{ U_n^{\left(\alpha;\, \substack{\beta,\, \gamma \\ a,\, b}\right)} \right\}$$

of the sequence $\{s_n\}$ defined in Theorem 2.1 is called the

$$J\left(\alpha;\, \begin{matrix} \beta, & \gamma \\ a, & b \end{matrix}\right)$$

transform of $\{s_n\}$.

The transformation

$$J\left(\alpha;\, \begin{matrix} \beta, & \gamma \\ a, & b \end{matrix}\right)$$

is, clearly, regular whenever $\alpha \geqq -1$.

3. **Some properties of Mellin transforms.** Given a function $f(t)$ of bounded variation for $0 \leqq t \leqq 1$, then the function $F(z)$ defined by $F(z) = \int_0^1 t^z df(t)$ is called the Mellin transform of the function $f(t)$. A result concerning Mellin transforms, which we use later, is

Theorem C (Pitt's theorem). *Let the function $T(z)$ be a Mellin transform of the function $\alpha(t)$ where $\alpha(t)$ is of bounded variation in $\langle 0, 1\rangle$ satisfying $\alpha(0) = \alpha(+0) = 0$; $\alpha(1) = 1$. If $|T(z)| \geqq d > 0$ for* Re $z \geqq 0$ *then $\{T(z)\}^{-1}$ is also a Mellin transform of some function $\beta(t)$ of bounded variation in $\langle 0, 1\rangle$ satisfying $\beta(0) = \beta(+0) = 0$; $\beta(1) = 1$.*

For a proof of Theorem C and a generalization of it see [**3**, pp. 178–179].

The main result of this section is

Theorem 3.1. *Let a, b, α and β be real numbers satisfying $a + b = 1$, $0 < \alpha < \beta$, $0 \leqq a$ and $a\alpha + b\beta > 0$. Then the sequence $\{\mu_n\}$, defined by $\mu_0 = 1$; $\mu_n = \alpha \cdot [a\alpha + b\beta]^{-1} \cdot [1 - (n+1)^{-\alpha}] \cdot [1 - a(n+1)^{-\alpha} - b(n+1)^{-\beta}]^{-1}$, $n > 0$, is a regular moment sequence.*

In the proof of Theorem 3.1 we use two auxiliary propositions, Lemma 3.1 and Lemma 3.2; the first proposition

Lemma 3.1. *Let a, b, α and β be four real numbers satisfying $a + b = 1$, $0 < \alpha < \beta$, $0 \leqq a$ and $a\alpha + b\beta > 0$. Then the function $T(z)$ defined by $T(z) = (a\alpha + b\beta) \cdot \alpha^{-1} \cdot [1 - (z+1)^{-\alpha}] \cdot [1 - a(z+1)^{-\alpha} - b(z+1)^{-\beta}]$, is a Mellin transform of a function $\gamma(t)$ of bounded variation on $\langle 0, 1\rangle$ satisfying $\gamma(0) = \gamma(+0) = 0$; $\gamma(1) = 1$.*

follows simply from

LEMMA A. *For* $0<\alpha, \beta$ *the function* $T(z)$ *defined by*

$$T(z) = \frac{\beta}{\alpha} \cdot \frac{1-(z+1)^{-\alpha}}{1-(z+1)^{-\beta}}$$

$((z+1)^{-\alpha}=e^{-\alpha\cdot\log(z+1)}$, *where we choose for* $\log(z+1)$ *its principal branch) is a Mellin transform of a function* $\alpha(t)$, *of bounded variation in* $\langle 0, 1\rangle$, *satisfying* $\alpha(0)=\alpha(+0)=0$; $\alpha(1)=1$.

For a proof of Lemma A see [1, Lemma 9.1].

It is easy to prove the second auxiliary proposition

LEMMA 3.2. *With the suppositions of Lemma* 3.1 *on* a, b, α, *and* β *and for the function* $T(z)$ *defined there, there exists a positive number* g *such that for* $\operatorname{Re} z \geqq 0$, $|T(z)| \geqq g > 0$.

The proof of Theorem 3.1 follows immediately by combining Lemma 3.2 with Theorem C.

4. Further properties of $J\left(\alpha; \begin{matrix}\beta, \gamma\\ a, b\end{matrix}\right)$ methods of summability.

In this section we prove a property of the

$$J\left(\alpha; \begin{matrix}\beta, \gamma\\ a, b\end{matrix}\right)$$

method of summability which is formulated in

THEOREM 4.1. *For all real numbers* a, b, α, β *and* γ *satisfying* $a+b=1$, $\alpha<\beta<\gamma$, $0\leqq a$ *and* $a\alpha+b\beta>0$ *the*

$$J\left(\alpha; \begin{matrix}\beta, \gamma\\ a, b\end{matrix}\right)$$

method of summability is equivalent to the $(H, \alpha+1)$ *method of summability.*

PROOF. Since both the transformations

$$J\left(\alpha; \begin{matrix}\beta, \gamma\\ a, b\end{matrix}\right)$$

and $J^{(\alpha,\beta)}$ are Hausdorff transformations and, by Theorem B, the $J^{(\alpha,\beta)}$ and $(H, \alpha+1)$ methods of summability are equivalent, it is enough (by a well known theorem of Hausdorff) to show that the following two sequences $\{\mu_n\}$ and $\{\lambda_n\}$, where

$$\mu_0 = 1;\ \mu_n = \frac{\beta - \alpha}{a(\beta-\alpha) + b(\gamma-\alpha)} \cdot \frac{1 - (n+1)^{-(\beta-\alpha)}}{1 - a(n+1)^{-(\beta-\alpha)} - b(n+1)^{-(\gamma-\alpha)}},$$
$$n > 0,$$

$$\lambda_0 = 1;\ \lambda_n = \frac{a(\beta-\alpha) + b(\gamma-\alpha)}{\beta - \alpha} \cdot \frac{1 - a(n+1)^{-(\beta-\alpha)} - b(n+1)^{-(\gamma-\alpha)}}{1 - (n+1)^{-(\beta-\alpha)}},$$
$$n > 0,$$

are regular moment sequences; which is precisely the content of Theorem 3.1 and Lemma 3.1, respectively. Q.E.D.

5. **On products of summability methods.** The following result on the product of $A^{(\alpha)}$ and regular Hausdorff methods of summability was proved by me in [1] (the special case $\alpha=0$ of this proposition is due to O. Szász. See [4]).

THEOREM D. *If, for some $\alpha > -1$, $\{s_n\}$ is summable $A^{(\alpha)}$ to s and $\{t_n\}$ is any regular Hausdorff transform of $\{s_n\}$ then $\{t_n\}$ is also summable $A^{(\alpha)}$ to s.*

6. **Two tauberian properties of the $A^{(\alpha)}$ methods of summability.** The two tauberian theorems for $A^{(\alpha)}$ methods of summability, formulated below, are known (compare [2, Theorem 3.1 and its proof]).

THEOREM E. *If, for some $\alpha > -1$, $\{s_n\}$ is summable $A^{(\alpha)}$ to s and $s_n - s_{n-1} = O(1/n)$, $n \to \infty$, then $\{s_n\}$ is summable $(H, -1+\epsilon)$ to s for each $\epsilon > 0$.*

THEOREM F. *If, for some $\alpha > -1$, $\{s_n\}$ is summable $A^{(\alpha)}$ to s and $s_n - s_{n-1} = O_L(1/n)$, $n \to \infty$, then $\{s_n\}$ is convergent to s.*

7. **A tauberian theorem for Hölder summability.** The tauberian theorem, for Hölder summability, stated below is also interesting in itself.

THEOREM 7.1. *Let $\{s_n\}$ be summable Hölder to s. Then, given a real number α arbitrarily a necessary and sufficient condition for $\{s_n\}$ to be summable (H, α) is that for a system of $2n$ real numbers $a_1, a_2, \cdots, a_n, \alpha_1, \alpha_2, \cdots, \alpha_n$ satisfying $\alpha < \alpha_1 < \cdots < \alpha_n$ and $a_1 + a_2 + \cdots + a_n = 1$ holds*

$$\lim_{m\to\infty} \{h_m^{(\alpha)} - [a_1 h_m^{(\alpha_1)} + a_2 h_m^{(\alpha_2)} + \cdots + a_n h_m^{(\alpha_n)}]\} = 0. \tag{7.1}$$

An argument similar to that used in the proof of Theorem 2.1 of my paper [1] establishes Theorem 7.1 as well as a similar result for Cesàro summability. I do not state the last result explicitly here.

8. Tauberian theorems for the $A^{(\alpha)}$ methods of summability. We can prove now the generalizations just mentioned in §1.

THEOREM 8.1. *Let a, b, α, β and γ be five real numbers satisfying $a+b=1$, $\alpha<\beta<\gamma$, $0\leqq a$ and $a(\beta-\alpha)+b(\gamma-\alpha)>0$. Then necessary and sufficient conditions for the (H, α) summability of $\{s_n\}$ are*

$$\lim_{n\to\infty} \{h_n^{(\alpha)} - [ah_n^{(\beta)} + bh_n^{(\gamma)}]\} = 0 \tag{8.1}$$

and that for some $\delta>-1$ $\{s_n\}$ is summable $A^{(\delta)}$.

PROOF. It is easy to see that (8.1) and the $A^{(\delta)}$ summability are necessary for the summability considered. Now we show that (8.1) together with the $A^{(\delta)}$ summability are sufficient for the same purpose. First we suppose that $\alpha\geqq 0$. By Theorem D and Theorem 2.1 we have for

$$\{U_n^{(\alpha;\,{\beta,\,\gamma \atop a,\,b})}\},$$

the

$$J\left(\alpha;\, {\beta,\ \gamma \atop a,\ b}\right)$$

transform of $\{s_n\}$,

$$\lim_{x\uparrow 1} (1-x)^{\delta+1} \sum_{n=0}^{\infty} \binom{n+\delta}{n} \cdot U_n^{(\alpha;\,{\beta,\,\gamma \atop a,\,b})} x^n = s.$$

From Theorem 2.1 and Theorem E we infer, by (8.1), that $\{s_n\}$ is summable

$$J\left(\alpha;\, {\beta,\ \gamma \atop a,\ b}\right)$$

to s. Hence, by Theorem 4.1, $\{s_n\}$ is summable $(H, \alpha+1)$ to s; thus Theorems 7.1 and 8.1 yield $\lim_{n\to\infty} h_n^{(\alpha)}=s$. Our theorem is proved in the case $\alpha\geqq 0$. If, now, $\alpha<0$ then there exists a positive integer d such that $\alpha+d>0$, hence, by taking the (H, d) transform of the sequence $\{h_n^{(\alpha)}-[ah_n^{(\beta)}+bh_n^{(\gamma)}]\}$, possessing the limit zero, we obtain $h_n^{(\alpha+d)}-[ah_n^{(\beta+d)}+bh_n^{(\gamma+d)}]\to 0$, $n\to\infty$, and the last relation shows, by the first part of the proof, that $\{s_n\}$ is summable Hölder; whence the rest of the proof follows now by Theorem 7.1. Q.E.D.

Theorem 8.1 and the other results of this section have already been proved in [1] in the case $a=1$.

An argument similar to that used in the proof of Theorem 8.1 (but exploiting now Theorem E entirely and also using Theorem F as in [1, proof of Theorem 6.2 and proof of Theorem 6.3]) yields

THEOREM 8.2. *Let a, b, α, β and γ be five real numbers satisfying $a+b=1$, $\alpha<\beta<\gamma$, $0\leqq a$ and $a(\beta-\alpha)+b(\gamma-\alpha)>0$. If, for some real number $\delta>-1$, $\{s_n\}$ is summable $A^{(\delta)}$ and* (i) $h_n^{(\alpha)}-[ah_n^{(\beta)}+bh_n^{(\gamma)}]=O_L(1)$, *as $n\to\infty$, then $\{s_n\}$ is summable $(H, \alpha+1)$ to s. If, further,* (ii) $h_n^{(\alpha)}-[ah_n^{(\beta)}+bh_n^{(\gamma)}]=O(1)$, *as $n\to\infty$, then $\{s_n\}$ is summable $(H, -1+\alpha+\epsilon)$ to s for each $\epsilon>0$.*

The argument used in the proof of Theorem 5.2 of [1] together with Theorem 3.1 of the present paper enable us to obtain

THEOREM 8.3. *Let the nine real numbers a, a', b, b', α, β, β', γ and γ' satisfy $\alpha<\beta<\gamma$, $\alpha<\beta'<\gamma'$, $a+b=a'+b'=1$, $a(\beta-\alpha)+b(\gamma-\alpha)>0$ and $a'(\beta'-\alpha)+b'(\gamma'-\alpha)>0$. If*

$$\lim_{n\to\infty}\frac{h_n^{(\alpha)}-[ah_n^{(\beta)}+bh_n^{(\gamma)}]}{a(\beta-\alpha)+b(\gamma-\alpha)}=l$$

then also

$$\lim_{n\to\infty}\frac{h_n^{(\alpha)}-[a'h_n^{(\beta')}+b'h_n^{(\gamma')}]}{a'(\beta'-\alpha)+b'(\gamma'-\alpha)}=l.$$

REMARK. Combining arguments of [1, §7], with propositions of this note we may obtain results for the Borel method of summability, similar to those of this section, which generalize those of [1, §7]. I do not state these results explicitly here.

9. **Further properties of the Hölder transformations.** Given $2n+1$ real numbers $a_1, a_2, \cdots, a_n, \alpha, \alpha_1, \alpha_2, \cdots, \alpha_n$ satisfying $\alpha<\alpha_1<\alpha_2<\cdots<\alpha_n$ and $a_1+a_2+\cdots+\alpha_n=1$ we call the sequence

$$\left\{U_m^{\left(\alpha;\,{\alpha_1,\cdots,\alpha_n \atop a_1,\cdots,a_n}\right)}\right\}$$

defined by

$$U_0^{\left({\alpha_1,\cdots,\alpha_n \atop a_1,\cdots,a_n}\right)}=h_0^{(\alpha)};\ U_m^{\left(\alpha;\,{\alpha_1,\cdots,\alpha_n \atop a_1,\cdots,a_n}\right)}=h_0^{(\alpha)}$$
$$+\sum_{v=1}^{m}\frac{h_v^{(\alpha)}-[a_1h_v^{(\alpha_1)}+\cdots+a_nh_v^{(\alpha_n)}]}{[a_1(\alpha_1-\alpha)+\cdots+a_n(\alpha_n-\alpha)]v},\quad n>0,$$

the

$$J\left(\alpha; \begin{matrix} \alpha_1, \cdots, \alpha_n \\ a_1, \cdots, a_n \end{matrix}\right)$$

transform of the sequence $\{s_n\}$. Using results of the previous sections it is easy to show that the

$$J\left(\alpha; \begin{matrix} \alpha_1, \cdots, \alpha_n \\ a_1, \cdots, a_n \end{matrix}\right)$$

transformation is a Hausdorff transformation generated by the sequence $\{\mu_m\}$, where

$$\mu_0 = 1; \quad \mu_m = \frac{(m+1)^{-\alpha} - [a_1(m+1)^{-\alpha_1} + \cdots + a_n(m+1)^{-\alpha_n}]}{[a_1(\alpha_1 - \alpha) + \cdots + a_n(\alpha_n - \alpha)]\cdot m}, \qquad m > 0.$$

Taking this into consideration we can prove now one of the main results of this section, namely

THEOREM 9.1. *Given $2n+1$ real numbers $a_1, a_2, \cdots, a_n, \alpha, \alpha_1, \alpha_2, \cdots, \alpha_n$ satisfying $\alpha<\alpha_1<\alpha_2< \cdots <\alpha_n$, $0<\min(a_1, \cdots, a_n)$ and $a_1+a_2+ \cdots +a_n=1$, the*

$$J\left(\alpha; \begin{matrix} \alpha_1, \cdots, \alpha_n \\ a_1, \cdots, a_n \end{matrix}\right)$$

method of summability is equivalent to the $(H, \alpha+1)$ method of summability; thus the

$$J\left(\alpha; \begin{matrix} \alpha_1, \cdots, \alpha_n \\ a_1, \cdots, a_n \end{matrix}\right)$$

method of summability is regular if $\alpha \geqq -1$.

The proof of Theorem 9.1 is the same as that of Theorem 4.1 but now we have to use a simple modification of Lemma 3.2.

Now we mention a result which is more general than Theorem 9.1. This generalization is obtained by using the following argument.

Let α, β and γ be three real numbers satisfying $\alpha<\beta<\gamma$. If $a(u)$ is a nondecreasing and bounded function in $\beta \leqq u \leqq \gamma$ which satisfies $a(\gamma)-a(\beta)=1$ then call

$$\left\{ U_n^{\left(\alpha; \begin{smallmatrix} \beta, \gamma \\ a(u) \end{smallmatrix}\right)} \right\}$$

defined by

$$U_0^{\left(\alpha;\,\substack{\beta,\,\gamma\\ a(u)}\right)} = h_0^{(\alpha)}; \quad U_n^{\left(\alpha;\,\substack{\beta,\,\gamma\\ a(u)}\right)} = h_0^{(\alpha)} + \sum_{v=1}^{n} \frac{h_v^{(\alpha)} - \int_\beta^\gamma h_n^{(t)}\,da(t)}{v\cdot\int_\beta^\gamma (t-\alpha)\,da(t)}, \quad n > 0,$$

the

$$J\begin{pmatrix}\alpha; & \beta,\,\gamma\\ & a(u)\end{pmatrix}$$

transform of $\{s_n\}$. It is easy to see that this transformation is a Hausdorff transformation generated by the sequence $\{\mu_n\}$ where

$$\mu_0 = 1; \qquad \mu_n = \frac{(n+1)^{-\alpha} - \int_\beta^\gamma (n+1)^{-t}\,da(t)}{n\cdot\int_\beta^\gamma (t-\alpha)\,da(t)}, \qquad n > 0.$$

In the same way that Theorem 4.1 was proved we may prove

THEOREM 9.2. *If* (i) α, β *and* γ *are three real numbers satisfying* $\alpha<\beta<\gamma$ *and* (ii) $a(u)$ *is a nondecreasing and bounded function in* $\beta\leqq u\leqq\gamma$ *satisfying* $a(\gamma)-a(\beta)=1$, *then the*

$$J\begin{pmatrix}\alpha; & \beta,\,\gamma\\ & a(u)\end{pmatrix}$$

method of summability is equivalent to the $(H, \alpha+1)$ *method; thus the*

$$J\begin{pmatrix}\alpha; & \beta,\,\gamma\\ & a(u)\end{pmatrix}$$

method of summability is regular if $\alpha\geqq-1$.

It is easy to see that if in Theorem 8.1, Theorem 8.2 and Theorem 8.3 we replace expressions of the form $h_m^{(\alpha)}-[ah_m^{(\beta)}+bh_m^{(\gamma)}]$ by expressions of the form $h_m^{(\alpha)}-\int_\beta^\gamma h_m^{(u)}da(u)$, where $a(u)$ is a nondecreasing and bounded function in $\beta\leqq u\leqq\gamma$ satisfying $a(\gamma)-a(\beta)=1$, and expressions of the form $a(\beta-\alpha)+b(\gamma-\alpha)$ by expressions of the form $\int_\beta^\gamma(u-\alpha)da(u)$, then the modified conclusions of these modified theorems remain valid.

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ON A THEOREM OF J. L. WALSH

ARYEH DVORETZKY

1. In a recent paper [1] J. L. Walsh proved, among other results, the following theorem:

Let the functions $f_n(x)$ $(n=1, 2, \cdots)$ and $f(x)$ be p times differentiable in the interval $a<x<b$ and let $f_n(x)$ converge to $f(x)$ in this interval. Then, given any point $x_0\in(a, b)$ there exists a sequence of points $x_n\in(a, b)$ such that

$$\lim_{n=\infty} x_n = x_0, \qquad \lim_{n=\infty} f_n^{(p)}(x_n) = f^{(p)}(x_0). \tag{1}$$

The main purpose of this short note is to show that "in general" there exists a sequence x_n satisfying the first condition of (1) and for which $f_n^{(p)}(x_n)=f^{(p)}(x_0)$ for all sufficiently large n; and when this does not occur then for the corresponding n not only (1) holds but $f_n^{(p)}(x)$ is close, in a sense which will be made precise, to $f^{(p)}(x_0)$ in the neighborhood of x_0. While doing this we shall replace the convergence assumption by a considerably weaker one.

2. THEOREM. *Let $f(x)$ and $f_n(x)$ $(n=1, 2, \cdots)$ be p times differentiable in the interval $a<x<b$ and let*

$$\lim_{n=\infty} \operatorname*{Inf}_{x\in I, y\in I} \left| f_n(y) - f(x) \right| = 0 \tag{2}$$

for every open sub-interval I of (a, b). Then, given any point $x_0\in(a, b)$, the sequence $N=\{n\}$ can be written as a union of two (not necessarily both infinite) sequences $N_1=\{n_1\}$ and $N_2=\{n_2\}$ in such a way that

Received by the editors March 28, 1955 and, in revised form, May 23, 1955.