$$\sum_{|n|=n_{k(\mu)}+1}^{n_{k(\mu)+1}} (\log |n|) |c_n|^2 \leqq \epsilon_{k(\mu)} \log n_{k(\mu)+1} \leqq 2D_\mu.$$

We choose the corresponding Fourier coefficients to define as before a new function $g(x)$, whose Fourier series converges almost everywhere since $\sum_{\mu=1}^{\infty} D_\mu < \infty$. Now we define $\{m_\nu\}$ to take on the values m, $n_{k(\mu)} < m \leqq n_{k(\mu)+1}$ for each μ. Since the sequence $\{n_{k(\mu)}\}$ is lacunary, the almost everywhere convergence of $s_{m_\nu}(x; f)$ to $f(x)$ follows as before. For the sequence $\{m_\nu\}$,

$$\frac{\sigma(n_{k(\mu)+1})}{n_{k(\mu)+1}} \geqq \frac{n_{k(\mu)+1} - n_{k(\mu)}}{n_{k(\mu)+1}} = 1 - \frac{1}{\lambda_\mu}$$

for each μ. Since the limit of the right side is 1, the theorem is proved.

Reference

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ON THE LOGARITHMIC MEAN OF THE DERIVED CONJUGATE SERIES OF A FOURIER SERIES

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1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π and let

$$(1.1)\quad f(t) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{1}^{\infty} A_n(t).$$

The differentiated conjugate series of (1.1) at $t=x$ is

$$(1.2)\qquad -\sum_{1}^{\infty} x(a_n \cos nx + b_n \sin nx) = -\sum_{1}^{\infty} nA_n(x).$$

We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x), \qquad h(t) = \frac{\phi(t)}{4 \sin \frac{1}{2}t} - d,$$

where d is a function of x.

Let S_n, t_n, and σ_n be the nth partial sum, the first Cesàro mean, and the first logarithmic mean of the series (1.2) respectively. The

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object of the present note is to prove the following

THEOREM. *If*

$$\int_t^\pi \frac{|h(u)|}{u}\,du = o\left(\log\frac{1}{t}\right) \text{ as } t \to 0, \tag{1.3}$$

then

$$\lim_{n\to\infty} (\sigma_{2^n} - \sigma_n) = \frac{d}{\pi}\log 2.$$

This is analogous to results for the conjugate series of (1.1) contained in [**3**; **1**]. Plainly condition (1.3) implies that $h(t)$ is integrable L in $(0, \pi)$. In a previous note [**2**] it was proved that, under condition (1.3),

$$t_n \sim \frac{2d}{\pi}\log n,$$

which in turn implies that

$$\sigma_n \sim \frac{d}{\pi}\log n.$$

Justification for this statement is provided by the identity

$$\sigma_n = \frac{t_n}{\log n} + \frac{1}{\log n}\sum_{k=1}^{n}\frac{t_{k-1}}{k} \qquad (t_0 = 0).$$

It is enough to prove the theorem for the special case in which $d=0$. To justify this assertion, we may restrict ourselves to the special case in which $x=0$ without any loss of generality.

Consider first the case in which

$$f(t) = \frac{1}{2}\pi t - \frac{1}{4}t^2 \sim \frac{1}{6}\pi^2 - \sum_{n=1}^{\infty}\frac{\cos nt}{n^2}\cdot$$

In this special case both the hypothesis and the conclusion of the Theorem remain true. In the general case, write

$$f(t) = f_1(t) - \frac{2d}{\pi}\left(\frac{1}{2}\pi t - \frac{1}{4}t^2\right),$$

with d corresponding to $f_1(t)$ taken as 0.

2. **Proof of the Theorem.** We have

$$
\begin{aligned}
rA_r(x) &= \frac{1}{\pi}\int_0^\pi \phi(t) r \cos rt dt \\
&= \frac{4}{\pi}\int_0^\pi h(t) \sin \frac{1}{2} t \frac{d}{dt} (\sin rt) dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S_n &= -\frac{4}{\pi}\int_0^\pi h(t) \sin \frac{1}{2} t \frac{d}{dt}\left(\sum_{r=1}^{n} \sin rt\right) dt \\
&= -\frac{2}{\pi}\int_0^\pi h(t) \sin \frac{1}{2} t \frac{d}{dt}\left\{\cot \frac{1}{2} t(1-\cos nt) + \sin nt\right\} dt.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\sigma_n &= \frac{1}{\log n}\sum_{k=1}^{n} \frac{S_k}{k} \\
&= -\frac{2}{\pi \log n}\int_0^\pi h(t) \sin \frac{1}{2} t \frac{d}{dt}\left\{\cot \frac{1}{2} t \sum_{k=1}^{n} \frac{1-\cos kt}{k}\right. \\
&\qquad\qquad \left. + \sum_{k=1}^{n} \frac{\sin kt}{k}\right\} dt.
\end{aligned}
$$

It follows that

$$
\sigma_{2^n} - \sigma_n = -\frac{2}{\pi \log n \log 2n}\int_0^\pi h(t) K_n(t) dt, \tag{2.1}
$$

where

$$
\begin{aligned}
K_n(t) = {} & \sin \frac{1}{2} t\left[\log n \sum_{1}^{2^n} \cos kt - \log 2n \sum_{1}^{n} \cos kt\right] \\
& + \sin \frac{1}{2} t \frac{d}{dt}\left\{\cot \frac{1}{2} t\left[\log n \sum_{1}^{2^n} \frac{1-\cos kt}{k}\right.\right. \\
& \qquad\qquad \left.\left. - \log 2n \sum_{1}^{n} \frac{1-\cos kt}{k}\right]\right\}.
\end{aligned}
$$

By means of the elementary relations

$$
\sum_{1}^{m} \cos kt = O\left(\frac{1}{\sin (t/2)}\right), \quad \sum_{1}^{m} \sin kt = \frac{1}{2} \cot \frac{1}{2} t(1-\cos mt) + O(1)
$$

we obtain

$$K_n(t) = O(\log n) - \frac{1}{2}\operatorname{cosec}\frac{1}{2}t\left[\log n \sum_1^{2n} \frac{1-\cos kt}{k} - \log 2n \sum_1^{n} \frac{1-\cos kt}{k}\right] \tag{2.2}$$
$$+ \frac{1}{2}\cos\frac{1}{2}t\cot\frac{1}{2}t[\log n(1-\cos 2nt) - \log 2n(1-\cos nt)];$$

the first square bracket may be also written in the form

$$-\log 2\sum_1^n \frac{1-\cos kt}{k} + \log n \sum_{n+1}^{2n} \frac{1-\cos kt}{k}.$$

We now use the result [1, proof of Theorem C] that (1.3) implies

$$\int_0^\pi h(t)\cot\frac{1}{2}t(1-\cos nt)dt = o(\log n).$$

Since $h(t)$ is integrable, this is also true with $h(t)\cot(t/2)$ replaced by $h(t)\operatorname{cosec}(t/2)$, or by $h(t)\cos(t/2)\cot(t/2)$. From (2.1) and (2.2) we obtain

$$\sigma_{2^n} - \sigma_n = o(1) + O\left(\log^{-2} n \sum_1^n \frac{\log k}{k} + \log^{-1} n \sum_{n+1}^{2n} \frac{\log k}{k}\right) = o(1).$$

This completes the proof of the theorem.

Finally we must express our thanks to the referee for some suggestions which improved the presentation and simplified the proof.

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