ON FINITE PROJECTIVE GAMES

MOSES RICHARDSON[1]

1. **Preliminaries on simple games.** Let $N=\{1, 2, \cdots, n\}$ be a finite set of n elements termed *players*. Let $\mathcal{N}$ be the class of all subsets S of N; the elements S of $\mathcal{N}$ are termed *coalitions*. If $\mathcal{S}\subset\mathcal{N}$, let $\mathcal{S}^+$ denote the class of all supersets of elements of $\mathcal{S}$, and $\mathcal{S}^*$ the class of all complements of elements of $\mathcal{S}$; in symbols, $\mathcal{S}^+=[X\in\mathcal{N} \mid X\supset S$ for some $S\in\mathcal{S}]$, $\mathcal{S}^*=[X\in\mathcal{N} \mid N-X\in\mathcal{S}]$. By a *simple game* is meant an ordered pair $G=(N, \mathcal{W})$ where $\mathcal{W}\subset\mathcal{N}$ satisfies (1) $\mathcal{W}=\mathcal{W}^+$, (2) $\mathcal{W}\cap\mathcal{W}^*=0$. The elements of $\mathcal{W}$ are termed *winning coalitions*. The elements of $\mathcal{L}=\mathcal{N}-\mathcal{W}$ are termed *losing coalitions*. The elements of $\mathcal{B}=\mathcal{L}\cap\mathcal{L}^*$ are termed *blocking coalitions*. A simple game[2] is termed *strong* if $\mathcal{B}=0$. A simple game may be defined by specifying the class $\mathcal{W}^m\subset\mathcal{W}$ of minimal winning coalitions. By an *imputation* is meant an ordered n-tuple of real numbers $x=(x_1, x_2, \cdots, x_n)$ such that[3] $x_i\geqq 0$ and $\sum_{i=1}^n x_i=1$. If $\mathcal{U}\subset\mathcal{N}$, let $\mathcal{U}^0=\mathcal{N}-(\mathcal{U}^+)^*$; $\mathcal{U}^0$ is the class of all coalitions which intersect every element of $\mathcal{U}$. If $\mathcal{U}=\mathcal{W}^m$ then $\mathcal{U}^0=\mathcal{L}^*=\mathcal{W}\cup\mathcal{B}$.

Suppose given a simple game $(N, \mathcal{W})$, a nonempty class $\mathcal{U}\subset\mathcal{W}$, and real numbers $a_1, a_2, \cdots, a_n$ such that

$$\text{(i)} \qquad \sum_{i\in S} a_i = 1 \text{ for } S\in\mathcal{U},$$

$$\text{(ii)} \qquad \sum_{i\in S} a_i > 1 \text{ for } S\in\mathcal{U}^0-\mathcal{U}.$$

Let $x^{(S)}$ denote the imputation of which the ith component is a_i if $i\in S$ and 0 otherwise. Then the finite set of imputations $X=[x^{(S)} \mid S\in\mathcal{U}]$ is termed a *simple solution* of the game $(N, \mathcal{W})$. If $\mathcal{U}=\mathcal{W}^m$ then X is termed a *main simple solution* (cf. [4], pp. 443–444).

2. **Finite projective games.** The following remarks stem from

Presented to the Society, April 16, 1955; received by the editors November 4, 1954 and, in revised form, June 19, 1955.

[1] This work was done for the Logistics Project sponsored by the Office of Naval Research in the Department of Mathematics, Princeton University.

[2] The original definition of simple games in von Neumann and Morgenstern [4] is such as to forbid the existence of blocking coalitions. Thus the simple games of [4] are our strong simple games. The definitions used here are due to Shapley [5].

[3] We use the (0, 1)-normalization. The precise relationship, not needed for reading this paper, between this normalization and that of [4] can be found explicitly in [2] or [3].

curiosity concerning a footnote in von Neumann and Morgenstern [4, p. 469, footnote 3], to the effect that finite projective geometries other than the seven-point one seem unsuitable for the "present purpose" of providing examples of simple games. The explanation of this statement is given by Theorem 1 below, in view of our footnote 2.

Consider the k-dimensional projective space $PG(k, p^n)$ whose field of coordinates is the Galois field $GF(p^n)$ where p is prime and n a positive integer.[4] We define a simple game based on this space as follows. The players shall be all the points of the k-space. Since no two winning coalitions can be complementary, it is essential to define the game by choosing the minimal winning coalitions so that any pair of them intersect. Since an l-space and an m-space in a projective k-space must intersect if $l+m\geqq k$, it is natural to select as minimal winning coalitions the linear subspaces of lowest dimension such that they all intersect pairwise. Thus, if k is even, $k=2h$, let the h-spaces be chosen; and if k is odd, $k=2h+1$, let the $(h+1)$ spaces be chosen. The simple game thus defined will be denoted also by $PG(k, p^n)$ and will be termed a *finite projective game.*

A blocking coalition is one which is not winning but which intersects every winning coalition. Clearly, if $k=2h+1$, all the h-spaces are blocking coalitions. These blocking coalitions have fewer members than the minimal winning coalitions, since the number of points in a q-space is $1+p^n+p^{2n}+\cdots+p^{qn}$ (cf. the corollary to Theorem 2 below). The remainder of this note confines itself to the simplest even-dimensional case, namely the finite projective plane games $PG(2, p^n)$. Here, the lines are the minimal winning coalitions, and a blocking coalition is a set of points containing no line but intersecting every line.

THEOREM 1. *The game $PG(2, p^n)$ is strong if $p^n=2$, and not strong if $p^n>2$. In particular, there exists a blocking coalition of $2p^n$ players if $p^n>2$.*

PROOF. Choose an arbitrary point b_1 as the first member of the proposed blocking coalition B. It intersects $1+p^n$ lines of the plane. Let l be one of these lines and let $b_2, b_3, \cdots, b_{p^n}$ be distinct points of l different from b_1. Each of the points b_i $(i=2, 3, \cdots, p^n)$ intersects p^n lines different from l. Together the set of points $(b_1, b_2, \cdots, b_{p^n})$ intersect $(1+p^n)+(p^n-1)p^n=p^{2n}+1$ lines. Let a be the (p^n+1)th point of l; a cannot be put into B. There are p^n lines left unintersected by the points so far put into B, all these lines containing a.

[4] Notation and basic facts concerning these finite projective spaces are due to Veblen and Bussey [8]. Another exposition can be found in Carmichael [1].

There are p^{2n} points of the plane not on l not yet used, p^n of them on each of the p^n lines through a just mentioned. We shall show that $p^n > 2$ is a necessary and sufficient condition that we can choose one point on each of these p^n lines to put into B so that no p^n+1 points of B are collinear. There are $p^n \cdot p^n \cdots p^n = (p^n)^{p^n} = p^{np^n}$ available p^n-tuples of points that can be chosen so as to intersect the remaining p^n lines. If $p^n > 2$, then $p^{np^n} > p^{2n}$. The points $b_1, b_2, \cdots, b_{p^n}$ have intersected only p^{2n} lines other than l. Therefore not all these p^{np^n} p^n-tuples can colline with any of the points $b_1, \cdots, b_{p^n}$. Hence there exists a p^n-tuple which together with the points $b_1, \cdots, b_{p^n}$ constitute a blocking coalition B of $2p^n$ members, if $p^n > 2$. If $p^n = 2$, this is impossible, cf. Figure 1. For the two remaining lines meeting at a contain 4 other points, say x and y on one line, and x' and y' on the other.

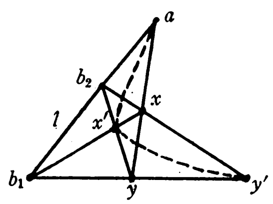

FIG. 1

Each of the 4 possible pairs xx', xy', $x'y$, or yy' collines with a used point of l. Therefore there exists no 4-person blocking coalition in this 7-point geometry $PG(2, 2)$, and in fact no blocking coalition at all. This completes the proof.

3. Simple solutions.

THEOREM 2. *If B is a blocking coalition in $PG(2, p^n)$, $p^n > 2$, then the number $|B|$ of members of B is greater than the number $1+p^n$ of points on a line.*

PROOF. *Case* 1. If p^n of the points of B are on some line l then they intersect $(p^n+1)+(p^n-1)p^n = p^{2n}+1$ lines, leaving p^n lines unintersected so far. A (p^n+1)th point not on l can then intersect only one new line. Since $1 < p^n$, not all lines are intersected by these p^n+1 points and $|B| \geqq p^n+2$.

Case 2. Suppose the maximum number of collinear points in B is less than p^n. Then any p^n points of B intersect fewer than $p^{2n}+1$ lines since at least one of them must intersect fewer than p^n new lines. Then more than p^n, i.e. at least p^n+1, lines are left unintersected. But the (p^n+1)th point of B cannot intersect more than p^n new lines.

Hence at least one line is still left unintersected and hence $|B| > p^n + 1$. This completes the proof.

COROLLARY. *Every two-dimensional finite projective game $PG(2, p^n)$ has a main simple solution.*[5]

PROOF. Let $a_i = 1/(p^n+1)$. There exists a simple solution consisting of one imputation for each line or minimal winning coalition S assigning a_i to $i \in S$ and 0 to $i \in -S$. Our Theorem 2, above, implies condition (ii) of the definition of simple solution, namely $\sum_{i \in S} a_i > 1$ for $S \in (\mathcal{W} \cup \mathcal{B}) - \mathcal{W}^m$, if $p^n > 2$. The case $p^n = 2$ is disposed of in [**4**, p. 469]. This completes the proof.[6]

4. **Blocking coalitions.** If min $|B|$ is the minimum number of members in a blocking coalition in $PG(2, p^n)$, $p^n > 2$, then we have established that $p^n + 2 \leqq \min |B| \leqq 2p^n$. It would be of interest to sharpen this result for $PG(2, p^n)$ by determining what min $|B|$ is exactly. The following fragmentary results bear on this problem.

THEOREM 3. *If a set S of points of $PG(2, p^n)$ contains fewer than $2p^n$ members and if p^n of the points of S are collinear but S contains no line, then the complementary set $-S$ contains at least one entire line.*

PROOF. Let the points $s_1, s_2, \cdots, s_{p^n}$ of S all lie on a line l. Then they intersect $p^{2n}+1$ lines. Any further point of $S \cap (-l)$ intersects just one line not intersected by $S \cap l$, namely the line determined by that point and $l \cap (-S)$, and hence intersects at most one new line. Hence if $|S| \leqq 2p^n - 1$, the number of intersected lines is not greater than $(p^{2n}+1) + (p^n - 1) \cdot 1 = p^{2n} + p^n$. This leaves at least one line not intersected by S, hence contained in $-S$.

COROLLARY. *The minimum number of elements in a blocking coalition of $PG(2, p^n)$, $p^n > 2$, which has p^n collinear points in it, is $2p^n$.*

However $2p^n$ is not in general the minimum number of elements in a blocking coalition of $PG(2, p^n)$. We show below that it is so for $PG(2, 3)$, but not for $PG(2, 4)$; in the latter case we exhibit a 7-point blocking coalition.

THEOREM 4. *In $PG(2, 3)$, the minimum number of elements in a blocking coalition is 6.*

[5] The author is indebted to L. S. Shapley for pointing out this corollary in conversation.

[6] We note parenthetically that $\sum_{i \in N} a_i = 1 + p^{2n}/(1+p^n) > 2$; compare (50:21) of p. 445 of [**4**] where the $2n$ is now replaced by 2 because of our use of the (0, 1)-normalization. See also [**5**]. Also parenthetically, it follows from Theorem 4 of [**3**] that $PG(2, p^n)$ is k-unstable for $p^n \leqq k < p^n + p^{2n}$ and k-stable for $1 \leqq k < p^n$.

Proof. We use the cyclic representation[7] of $PG(2, 3)$:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

in which the points are denoted by $0, 1, \cdots, 12$ and the lines consist of the points in the vertical columns.

Put an arbitrary point b_1 into the proposed blocking coalition B; it intersects 4 lines. Put any point $b_2 \neq b_1$ into B; it intersects 3 new lines. Put $b_3 \neq b_1, b_2$ into B; b_3 may be (A) on the line b_1b_2 or (B) not. In case (A), b_3 intersects 3 new lines with a cumulative total of 10 lines intersected. In case (B), b_3 intersects 2 new lines for a total of 9 lines intersected. Put $b_4 \neq b_1, b_2, b_3$ into B. In case (A), b_4 may not be collinear with b_1, b_2, b_3 because, if so, (b_1, b_2, b_3, b_4) is a line and hence a minimal winning coalition, not a blocking coalition; hence b_4 is not thus collinear with b_1, b_2, b_3 and therefore intersects one new line, for a total of 11 lines intersected. In case (B), b_4 may be: case (B1) on one of the lines b_1b_2, b_1b_3, or b_2b_3 in which case b_4 intersects 2 new lines for a total of 11; or case (B2) if b_4 is not on any of these 3 lines, then b_4 intersects one new line for a total of 10 lines intersected. Hence there exists no 4-person blocking coalition. Put $b_5 \neq b_1, b_2, b_3, b_4$ into B. In case (A), b_5 may not colline with b_1, b_2, b_3, as before; hence b_5 intersects at most one new line for a total of either 11 or 12. In case (B1), b_5 may not colline with b_1, b_2, b_4, say, for then (b_5, b_4, b_2, b_1) would be a line and hence not blocking, and hence yields a total of 12 at most. In case (B2), either: (i) b_5 is on one of the lines b_1b_2, b_1b_3, b_1b_4, b_2b_3, b_2b_4, b_3b_4 or on two of them, so that b_5 intersects 1 or 2 new lines, respectively, for a total of 11 or 12 lines intersected; or (ii) if b_5 is on none of these lines then it intersects no new line, for a total of 10 lines intersected. Hence there exists no 5-person blocking coalition. The set (0, 1, 5, 6, 7, 11) is a 6-person blocking coalition. This completes the proof.

In $PG(2, 4)$, we shall exhibit a 7-person blocking coalition. The cyclic representation of $PG(2, 4)$ is:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3
14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13
16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15.

[7] The existence of such a cyclic representation for all $PG(k, p^n)$ was established by J. Singer [6]. Another proof appears in E. Snapper [7].

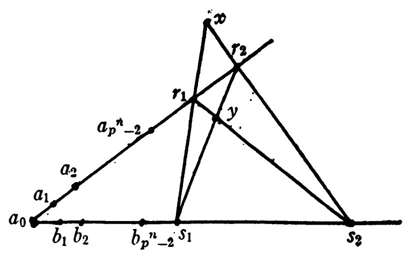

FIG. 2

Then the set (0, 1, 4, 5, 6, 15, 20) is a 7-person blocking coalition. This illustrates the following theorem.

THEOREM 5. *If $p^n > 3$, there exists in $PG(2, p^n)$ a blocking coalition of $2p^n - 1$ members.*

PROOF. Cf. Figure 2. Let $a_0, a_1, \cdots, a_{p^n-2}$ be distinct collinear points, and let $b_1, b_2, \cdots, b_{p^n-2}$ be distinct points collinear with a_0 but not on the line a_0a_1. These $2p^n-3$ points intersect $(p^n+1) + (p^n-2)p^n + (p^n-2)2 = p^{2n}+p^n-3$ lines. Let r_1, r_2 be the remaining points on a_0a_1 and s_1, s_2 the remaining points on a_0b_1. Then let $x = r_1s_1 \cap r_2s_2$ and $y = r_1s_2 \cap r_2s_1$. The points x, y intersect the four remaining lines $r_1s_1, r_2s_1, r_1s_2, r_2s_2$. Hence the set $B = [a_0, a_1, \cdots, a_{p^n-2}, b_1, b_2, \cdots, b_{p^n-2}, x, y]$ will constitute a blocking coalition of $2p^n-1$ points unless it contains a line. Now the collinear points $a_0, a_1, \cdots, a_{p^n-2}$ fall short of a line by two points, as do the collinear points $a_0, b_1, \cdots, b_{p^n-2}$. The points x and y are on neither of the two lines a_0a_1 and a_0b_1. Finally x and y colline with at most two points a_i, b_j of B, one from each of these two lines. (Note that x and y may colline with only one point a_0 of these two lines, since the diagonal points of a complete quadrangle colline if and only if $p=2$, but this does not affect our argument; cf. [8] or [1].) Hence the set B is a blocking coalition unless the set (x, y, a_i, b_j) contains a line, which can happen only if $p^n+1 \leqq 4$, or $p^n \leqq 3$. This completes the proof.

That Theorem 5 does not provide the minimum number of elements in a blocking coalition is shown by the next theorem.

THEOREM 6. *If d is a divisor of n, $1 \leqq d < n$, then there exists in $PG(2, p^n)$ a blocking coalition B with $2p^n - p^d + 1$ members.*

PROOF. In $PG(2, p^d)$, let $a_0, r_1, r_2, \cdots, r_{p^d}$ be the points of one line, let $a_0, s_1, s_2, \cdots, s_{p^d}$ be the points of a second line, and let $a_0, x_1, x_2, \cdots, x_{p^d}$ be the points of a third line through a_0. The set $X = [x_1, x_2, \cdots, x_{p^d}]$ clearly intersects all of the p^{2d} lines r_is_j.

Since d is a divisor of n and $1 \leqq d < n$, $PG(2, p^d)$ can be imbedded (cf. [8] or [1]) in $PG(2, p^n)$. Let L_r (L_s) be the line of $PG(2, p^n)$ containing the points r_i (s_j). Let $A = [a_1, a_2, \cdots, a_{p^n - p^d}]$ be the set of points of L_r not in $PG(2, p^d)$, and let $C = [c_1, c_2, \cdots, c_{p^n - p^d}]$ be the set of points of L_s not in $PG(2, p^d)$. Let $B = [a_0] \cup A \cup C \cup X$. Since $[a_0]$ intersects $1 + p^n$ lines of $PG(2, p^n)$, A intersects $(p^n - p^d)p^n$ new lines, C intersects $(p^n - p^d)p^d$ new lines, and X intersects p^{2d} new lines, it follows that B intersects all the lines of $PG(2, p^n)$. It is easily seen that B contains no line of $PG(2, p^n)$ and that $|B| = 2(p^n - p^d) + p^d + 1 = 2p^n - p^d + 1$. This completes the proof.

The following special case, communicated to the author by L. S. Shapley, shows that Theorem 6 does not provide a minimum.

THEOREM 7. *If $n = 2d$, then $PG(2, p^{2d})$ contains a blocking coalition with $1 + p^d + p^{2d}$ members.*

PROOF. The points of any $PG(2, p^d)$ imbedded in $PG(2, p^{2d})$ form such a coalition. For there are $1 + p^d + p^{2d}$ lines which are extensions of the lines of the subgeometry, and $p^{2d} - p^d$ additional lines through each point of the subgeometry, making a total of

$$1 + p^d + p^{2d} + (1 + p^d + p^{2d})(p^{2d} - p^d) = 1 + p^{2d} + p^{4d}.$$

Since this accounts for all the lines of $PG(2, p^{2d})$, the coalition blocks.

If $p^{2d} > 4$, this number $1 + p^d + p^{2d}$ is less than the number $2p^{2d} - p^d + 1$ provided by Theorem 6.

The problem of determining the number of points in a mininum blocking coalition remains open. In nongame-theoretic terms, the problem is to find the smallest number of points in a set which intersects every line but contains no entire line. Similar questions can be asked, of course, in the higher dimensional cases, in the non-Desarguesian geometries, and in those block designs in which every pair of distinguished sets intersect.

BIBLIOGRAPHY

1. R. D. Carmichael, *Introduction to the theory of groups of finite order*, Ginn, 1937.

2. D. B. Gillies, *Some theorems on n-person games*, Princeton University Doctoral Thesis, 1953, unpublished.

3. R. D. Luce, *A definition of stability for n-person games*, Ann. of Math. vol. 59 (1954) pp. 357–366.

4. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, 2d ed., Princeton, 1947.

5. L. S. Shapley, *Lectures on n-person games*, Princeton University Notes, to be published.

6. J. Singer, *A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 377–385.

7. E. Snapper, *Periodic transformations of affine and projective geometries*, Canadian Journal of Mathematics vol. 2 (1950) pp. 149–151.

8. O. Veblen and W. H. Bussey, *Finite projective geometries*, Trans. Amer. Math. Soc. vol. 7 (1906) pp. 241–259.

BROOKLYN COLLEGE

NOTE ON LINEAR FORMS

J. B. ROBERTS

1. There has been some interest in solutions to the equation

$$n = a_0x_0 + a_1x_1 + \cdots + a_sx_s \tag{*}$$

where the a_i are fixed positive integers with gcd$=1$ and the x_i are non-negative integers. In particular the question of finding the smallest n for which all greater integers have a solution has been investigated to some extent [1; 2]. It seems that the solution for $s=1$ has been known for some time but that the problem in general remains unsolved for $s>1$. In the paper of A. Brauer cited in the bibliography various upper bounds for the smallest n are given and the actual value of the smallest n is determined for the a_i consecutive integers. The main result of this paper is the determination of this smallest n when the a_i are in arithmetical progression.

2. Our investigation then is with the linear form

$$F = a_0x_0 + \cdots + a_sx_s.$$

Throughout this paragraph we assume $2 \leqq a_0$, gcd $a_i=1$ and $a_j=a_0+jd$. Thus the a_i are in arithmetical progression. Then we have the

THEOREM. *F represents all $n \geqq N$ where*

$$N = \left(\left[\frac{a_0-2}{s}\right]+1\right)\cdot a_0 + (d-1)(a_0-1)$$

with non-negative x_i and does not so represent $N-1$.

The proof of this result breaks down into a series of five lemmas.

LEMMA 1. *The only integers represented by F when $x_0+\cdots+x_s=m$ are ma_0, ma_0+d, $ma_0+2d, \cdots, ma_0+msd$.*

PROOF. F represents ma_0 for $x_0=m$, other $x_i=0$. If F represents ma_0+kd with $\sum_0^s x_i=m$ and $k<ms$ then $x_i>0$ for some $i<s$. In the representation of ma_0+kd replace $x_0, \cdots, x_i, x_{i+1}, \cdots, x_s$ by $x_0, \cdots, x_i-1, x_{i+1}+1, \cdots, x_s$. Now F represents $ma_0+(k+1)d$.

Received by the editors June 28, 1955.