Abel, Borel, Cesàro, Hölder and Hausdorff, Journal d'Analyse Mathématique vol. 3 (1953–1954) pp. 346–381.

2. ———, *On a converse of Abel's theorem*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 244–256.

3. W. W. Rogosinski, *On Hausdorff methods of summability*, Proc. Cambridge Philos. Soc. vol. 38 (1942) pp. 166–192.

4. O. Szász, *On the product of two summability methods*, Annales de la Société Polonaise de Mathématique vol. 25 (1952–1953) pp. 75–84.

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ON A THEOREM OF J. L. WALSH

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1. In a recent paper [1] J. L. Walsh proved, among other results, the following theorem:

Let the functions $f_n(x)$ $(n=1, 2, \cdots)$ and $f(x)$ be p times differentiable in the interval $a<x<b$ and let $f_n(x)$ converge to $f(x)$ in this interval. Then, given any point $x_0 \in (a, b)$ there exists a sequence of points $x_n \in (a, b)$ such that

$$\lim_{n=\infty} x_n = x_0, \qquad \lim_{n=\infty} f_n^{(p)}(x_n) = f^{(p)}(x_0). \tag{1}$$

The main purpose of this short note is to show that "in general" there exists a sequence x_n satisfying the first condition of (1) and for which $f_n^{(p)}(x_n) = f^{(p)}(x_0)$ for all sufficiently large n; and when this does not occur then for the corresponding n not only (1) holds but $f_n^{(p)}(x)$ is close, in a sense which will be made precise, to $f^{(p)}(x_0)$ in the neighborhood of x_0. While doing this we shall replace the convergence assumption by a considerably weaker one.

2. THEOREM. *Let $f(x)$ and $f_n(x)$ $(n=1, 2, \cdots)$ be p times differentiable in the interval $a<x<b$ and let*

$$\lim_{n=\infty} \operatorname*{Inf}_{x\in I, y\in I} \left| f_n(y) - f(x) \right| = 0 \tag{2}$$

for every open sub-interval I of (a, b). Then, given any point $x_0 \in (a, b)$, the sequence $N=\{n\}$ can be written as a union of two (not necessarily both infinite) sequences $N_1=\{n_1\}$ and $N_2=\{n_2\}$ in such a way that

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for every n_1 *there exists* $x_{n_1} \in (a, b)$ *for which*

$$f_{n_1}^{(p)}(x_{n_1}) = f^{(p)}(x_0) \tag{3}$$

and, in case N_1, *is infinite,*

$$\lim_{n_1=\infty} x_{n_1} = x_0; \tag{4}$$

while, if N_2 *is infinite, we have*

$$\limsup_{n_2=\infty} \int_{x_0-h}^{x_0+h} \left| f_{n_2}^{(p)}(x) - f^{(p)}(x_0) \right| dx = o(h) \tag{5}$$

as $0 < h \to 0$. *Moreover, if* x_0 *is not a local extremum point* (*in the wide sense*) *of* $f^{(p)}(x)$ *then the sequence* N_2 *may be taken as finite.*

Before proving the theorem we remark that assumption (2) is weaker not only than that of pointwise convergence, but also than those of convergence in measure or convergence on an everywhere dense set. On the other hand the conclusion (5) implies local convergence in measure in the sense that

$$\lim_{0<h\to 0} \limsup_{n_2=\infty} \mu(S_{n_2}, \epsilon)/2h = 1 \tag{6}$$

for every $\epsilon > 0$, where $\mu(S_{n_2}, \epsilon)$ is the Lebesgue measure of the subset of (x_0-h, x_0+h) where $\left| f_{n_2}^{(p)}(x) - f^{(p)}(x_0) \right| < \epsilon$.

Proof. We prove the assertion first for $p=1$. We can clearly split N into two disjoint sequences $N_1 = \{n_1\}$ and $N_2 = \{n_2\}$ so that N_1 has the properties stated in the theorem while for every n_2

$$f'_{n_2}(x) \neq f(x_0) \qquad \text{for } \alpha < x < \beta,$$

where α and β are fixed points satisfying $a < \alpha < x_0 < \beta < b$. Our aim is to prove that if N_2 is infinite then (5) holds. Since derivatives are Darboux functions we have for each n_2 either $f'_{n_2}(x) > f'(x_0)$ throughout (α, β) or $f'_{n_2}(x) < f(x_0)$ throughout (α, β). Splitting N_2 into two sets $N_3 = \{n_3\}$ and $N_4 = \{n_4\}$ according to these two cases, it is necessary to establish the conclusion for each of these sets. For definiteness sake we assume N_3 infinite and prove

$$\lim_{n_3=\infty} \int_{x_0-h}^{x_0+h} \left| f'_{n_3}(x) - f'(x_0) \right| dx = o(h). \tag{7}$$

Given $\epsilon > 0$ there exists h_0, $0 < h_0 < (1/2) \min (x_0 - \alpha, \beta - x_0)$ such that

$$|f(x_0+t)-f(x_0)-tf'(x_0)|<\epsilon|t| \tag{8}$$

whenever $|t|<2h_0$. Let h be any number satisfying $0<h\leqq h_0$; applying (2) to the intervals $(x_0-2h,\ x_0-h)$ and $(x_0+h,\ x_0+2h)$ we see that for all $n>m_0=m_0(\epsilon,h)$ there exist positive $\bar{\delta}_n$, $\bar{\delta}_n'$, δ_n, δ_n' all smaller than $\epsilon h/(1+|f'(x_0)|)$ and for which

$$\begin{aligned}&|f_n(x_0-h-\bar{\delta}_n)-f(x_0-h-\bar{\delta}_n')|<\epsilon h,\\ &|f_n(x_0+h+\delta_n)-f(x_0+h+\delta_n')|<\epsilon h.\end{aligned} \tag{9}$$

Putting $F_n(h)=f_n(x_0+h)-f(x_0)-hf'(x_0)$ and $F(h)=f(x_0+h)-f(x_0)-hf'(x_0)$, we obtain from (8) and (9)

$$|F_n(h+\delta_n)|<|F(h+\delta_n')|+|\delta_n-\delta_n'|\,|f'(x_0)|+\epsilon h<4\epsilon h$$

and, similarly, $|F_n(-h-\bar{\delta}_n)|<4\epsilon h$. But for all $n_3\in N_3$ we have $F_{n_3}'(t)=f_{n_3}'(x_0+t)-f'(x_0)>0$ for $|t|<2h$ and hence

$$\int_{-h}^{h}|f_{n_3}'(x_0+t)-f_{n_3}'(x_0)|\,dt$$

$$<\int_{-h-\bar{\delta}_n}^{h+\delta_n}F_{n_3}'(t)dt=F_{n_3}(h+\delta_n)-F_{n_3}(-h-\bar{\delta}_n)<8\epsilon h.$$

$\epsilon>0$ being arbitrary, this establishes (7).

Now assume that x_0 is not a local extremum of $f'(x)$. Then given any open interval (α,β) containing x_0 there exist in it two points x' and x'' for which $f'(x')>f'(x_0)>f'(x'')$. According to what has already been proved there exist, for all sufficiently large n, points x_n' and x_n'' in (α,β) for which $|f_n'(x_n')-f'(x')|<f'(x')-f'(x_0)$ and $|f_n'(x_n'')-f'(x'')|<f'(x_0)-f'(x'')$. In other words there exist x_n', x_n'' in (α,β) for which $f_n'(x_n')>f'(x_0)>f_n'(x_n'')$; using again the Darboux property, we see that there exists $x_n\in(\alpha,\beta)$ for which $f_n'(x_n)=f'(x_0)$. This completes the proof for $p=1$.

Let us now assume the theorem true for p and deduce it for $p+1$. In order to do this we merely have to remark that (5), which implies (6), implies in particular (2) with f_n and f replaced by $f_n^{(p)}$ and $f^{(p)}$ respectively. Thus applying the theorem for the case of first derivatives to $f_n^{(p)}$ and $f^{(p)}$ we obtain the required result. Q.E.D.

3. Remarks. (1) Hardly any change is needed in the proof in order to establish one-sided analogues of our theorem. Thus we may require that N_1 satisfy, in addition to (4) and (5), also $x_{n_1}>x_0$ provided we replace x_0-h by x_0 as the lower limit of integration in (5). This can

be extended to the end points of the interval. Thus the conclusion remains valid also for $x_0 = a$ provided $f(x)$ and $f_n(x)$ are defined also for $x = a$, $f(x)$ is p-times differentiable to the right there, the functions $f_n(x)$ are continuous to the right at $x = a$ (they need not be differentiable there), and $f_n(a) \to f(a)$ as $n \to \infty$.

(2) The theorem extends easily to functions of several variables.

(3) Condition (2) can be further weakened by assuming it to hold not for the functions $f_n(x)$ but for $g_n(x) = f_n(x) - P_n(x)$ where $P_n(x)$ are any polynomials of degree smaller than p.

(4) Simple examples show that, even assuming uniform convergence of $f_n(x)$ to $f(x)$, there does not exist any positive function $\phi(h)$ with $\phi(h)/h \to 0$ as $h \to 0$ having the property that (5) holds with $o(h)$ replaced by $o(\phi(h))$. Thus the $o(h)$ in (5) is "best possible." A similar remark applies to (6).

(5) A noninductive proof of the theorem can easily be given, and it actually yields somewhat sharper results for $p > 1$. We do not develop these results here since they are special cases of ones applying to more general operators, including among others those of fractional differentiation, which seem to merit a special study.

Reference

1. J. L. Walsh, *On continuity properties of derivatives of sequences of functions*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 69–75.

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