ON THE CONFORMAL MAPPING OF NEARLY-CIRCULAR DOMAINS[1]

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Let D be a simply-connected domain in the complex z-plane bounded by a smooth Jordan curve C, and denote by $F(z)=F(z, \xi)$ the analytic function which maps D conformally onto the unit disk and satisfies the additional conditions $F(\xi)=0$, $F'(\xi)>0$. It is well known [1; 4] that $F(z)$ and the Szegö kernel function $K(z, \xi)$ of D are connected by the relation

$$4\pi^2 K^2(z, \xi) = F'(z)F'(\xi).$$

Accordingly, the conformal mapping of D onto the unit disk can be carried out if the function $K(z, \xi)$ is known.

The only property of $K(z, \xi)$ which we shall use is the identity [1]

$$f(\xi) = \int_C [K(z, \xi)]^* f(z) ds, \tag{1}$$ [2]

which holds for all functions $f(z)$ of $L^2(D)$, that is, functions which are regular in D and for which $\int_C |f(z)|^2 ds < \infty$. $K(z, \xi)$ is itself in $L^2(D)$ and it is, moreover, the only function of $L^2(D)$ with the reproducing property (1). The dependence of $K(z, \xi)$ on ξ is shown by the identity $K(\xi, z) = [K(z, \xi)]^*$ which is easily derived from (1).

The object of this paper is the derivation of an approximation formula for $K(z, \xi)$ in the case in which D is a nearly circular domain whose boundary has the equation $r = 1 + \epsilon p(\theta)$ in polar coordinates, where ϵ is a small positive quantity. While it is not difficult to obtain such a formula with the help of devices of the type used in the derivation of Hadamard's formula for the variation of the Green's function [1; 2], the formula to be derived in this paper has the advantage of permitting the estimation of the maximum error committed in replacing $K(z, \xi)$ by its approximation.

The problem of estimating the error in approximation formulas for the conformal mapping of nearly-circular regions has been treated

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[2] Asterisks denote complex conjugates.

by a number of writers, notably Warschawski [5, and literature quoted there], whose work is based on the consideration of certain integral equations. The methods of the present paper, which utilize the properties of the Szegö kernel function, represent a different approach to this problem.

Our main result is

THEOREM I. *Let D be a domain bounded by a Jordan curve C with the equation*

$$r = 1 + \epsilon p(\theta), \qquad 0 \leqq \theta \leqq 2\pi, \tag{2}$$

where $p(\theta)$ and its derivative $p'(\theta)$ are continuous, $p(\theta)>0$, $p(2\pi)=p(0)$, $|p(\theta)| \leqq M$, $|p'(\theta)| \leqq M'$, and ϵ is a small positive quantity. If $K(z, \xi)$ denotes the Szegö kernel of D and $|z| \leqq \eta$, $|\xi| \leqq \eta$, where η is such that

$$\eta \leqq 1 - (2\epsilon)^{1/2}(M(M + M'))^{1/4}, \tag{3}$$

then

$$K(z, \xi) = \frac{1}{4\pi^2}\int_C \frac{ds}{(z - t)(\xi^* - t^*)} + \epsilon^2 R(z, \xi) \qquad (ds = |dt|) \tag{4}$$

where

$$|R(z, \xi)|^2 \leqq \rho(|z|)\rho(|\xi|), \tag{5}$$

and

$$\rho(\eta) = \frac{1}{2\pi(1 - \eta^2)}\left\{\frac{(2M + M')^2}{(1 - \eta)^4 - 4\epsilon^2 M(M + M')} + \frac{4M^2}{[1 - \eta(1 + \epsilon M)]^2}\right\}. \tag{5'}$$

For $z=\xi=0$, we have the particularly simple estimate

$$R(0, 0) \leqq \frac{1}{2\pi}(M^2 + M'^2). \tag{6}$$

We remark that the assumption $p(\theta)>0$ is no restriction, since any curve which is near the unit circumference can be trivially transformed into one for which this assumption holds.

For the proof of Theorem I we shall need the following two lemmas:

LEMMA I. *If $\Gamma(z, \xi)$ is defined by*

$$\Gamma(z, \xi) = \frac{1}{4\pi^2}\int_C \frac{ds}{(z - t)(\xi^* - t^*)}, \qquad z, \xi \in D, \tag{7}$$

then

$$\Gamma(\xi, \xi) \geqq K(\xi, \xi), \tag{8}$$

and

$$| \Gamma(z, \xi) - K(z, \xi) |^2 \leqq [\Gamma(z, z) - K(z, z)][\Gamma(\xi, \xi) - K(\xi, \xi)]. \tag{9}$$

PROOF. By the residue theorem,

$$\begin{aligned} &| \alpha |^2 K(z, z) + 2 \operatorname{Re} \{\alpha K(z, \xi)\} + K(\xi, \xi) \\ &\quad = \frac{1}{2\pi i} \int_C [\alpha^* K(t, z) + K(t, \xi)] \left[\frac{\alpha}{t - z} + \frac{1}{t - \xi} \right] dt, \end{aligned} \tag{10}$$

where α is an arbitrary constant. In view of (1), the left-hand side of (10) is equal to

$$\int_C | \alpha^* K(t, z) + K(t, \xi) |^2 ds.$$

Applying (7) and the Schwarz inequality, we obtain

$$\begin{aligned} &| \alpha |^2 K(z, z) + 2 \operatorname{Re} \{\alpha K(z, \xi)\} + K(\xi, \xi) \\ &\qquad \leqq | \alpha |^2 \Gamma(z, z) + 2 \operatorname{Re} \{\alpha \Gamma(z, \xi)\} + \Gamma(\xi, \xi), \end{aligned}$$

whence (8) and the discriminant inequality (9).

LEMMA II. *If C, r, and $p(\theta)$ have the same meaning as in Theorem* I, *then*

$$\begin{aligned} &\frac{1}{4\pi^2} \int_C \frac{ds}{r^2} - \left[\int_C ds \right]^{-1} \\ &\qquad \leqq \frac{\epsilon^2}{4\pi^2} \left[\int_0^{2\pi} p^2(\theta) \frac{d\theta}{r} + \frac{1}{2} \int_0^{2\pi} p'^2(\theta) \left(\frac{1}{r} + \frac{1}{r^3} \right) d\theta \right]. \end{aligned} \tag{11}$$

PROOF. It follows from (2) that

$$\begin{aligned} \int_C \frac{ds}{r^2} &= \int_0^{2\pi} \frac{1}{r} \left(1 + \frac{\epsilon^2 p'^2(\theta)}{r^2} \right)^{1/2} d\theta \leqq \int_0^{2\pi} \frac{1}{r} \left(1 + \frac{\epsilon^2 p'^2(\theta)}{2r^2} \right) d\theta \\ &= \int_0^{2\pi} \frac{d\theta}{1 + \epsilon p(\theta)} + \frac{\epsilon^2}{2} \int_0^{2\pi} \frac{p'^2(\theta)}{r^3} d\theta \\ &= 2\pi - \epsilon \int_0^{2\pi} p(\theta) d\theta + \epsilon^2 \int_0^{2\pi} \frac{p^2(\theta)}{r} d\theta + \frac{\epsilon^2}{2} \int_0^{2\pi} \frac{p'^2(\theta)}{r^3} d\theta, \end{aligned}$$

and

$$
\begin{aligned}
-\left[\int_C ds\right]^{-1} &= -\left[\int_0^{2\pi} r\left(1+\frac{\epsilon^2 p'^2(\theta)}{r^2}\right)^{1/2} d\theta\right]^{-1} \\
&\leqq -\left[\int_0^{2\pi}\left(1+\epsilon p(\theta)+\frac{\epsilon^2 p'^2(\theta)}{2r}\right)d\theta\right]^{-1} \\
&\leqq -\frac{1}{2\pi}+\frac{\epsilon}{4\pi^2}\int_0^{2\pi} p(\theta)d\theta+\frac{\epsilon^2}{8\pi^2}\int_0^{2\pi}\frac{p'^2(\theta)}{r}\,d\theta.
\end{aligned}
$$

Combining these two inequalities, we obtain (11).

We now enter upon the proof of Theorem I. If $|\xi| \leqq \eta$, where η satisfies the inequality (3), then it is easily confirmed that $|\xi|(1+\epsilon M) < 1$. The point $(\xi^*)^{-1}$ is therefore outside D, and the function $(1-\xi^* z)^{-1}$ belongs to $L^2(D)$. We may thus apply the identity (1). This yields

$$
\frac{1}{1-|\xi|^2} = \int_C [K(z,\xi)]^* \frac{ds_z}{1-\xi^* z},
$$

and therefore

$$
\begin{aligned}
\frac{1}{(1-|\xi|^2)^2} &\leqq \int_C |K(z,\xi)|^2 ds_z \int_C \frac{ds_z}{|1-\xi^* z|^2} \\
&= K(\xi,\xi)\int_C \frac{ds_z}{|1-\xi^* z|^2}.
\end{aligned}
$$

If $\Gamma(z,\xi)$ is the function defined in (7), it thus follows from (8) that

$$
\begin{aligned}
&0 \leqq \Gamma(\xi,\xi) - K(\xi,\xi) \\
&\leqq \frac{1}{4\pi^2}\int_C \frac{ds}{|z-\xi|^2} - \left[(1-|\xi|^2)^2\int_C \frac{ds}{|1-\xi^* z|^2}\right]^{-1}.
\end{aligned}
\tag{12}
$$

If we set $\xi=0$, the right-hand side of (12) reduces to the quantity estimated in Lemma II. In view of $r \geqq 1$, $|p(\theta)| \leqq M$, $|p'(\theta)| \leqq M'$, we may therefore conclude from (11) and (12) that

$$
0 \leqq \Gamma(0,0) - K(0,0) \leqq \frac{1}{2\pi}\,[M^2 + M'^2].
$$

Because of the definition (7), this proves (6).

To prove Theorem I in the general case, we have to find a similar estimate for the right-hand side of (12) if $\xi \neq 0$. To this end, we introduce the linear transformation

$$
w = (z-\xi)/(1-\xi^* z). \tag{13}
$$

Since $(1-\xi^*z)^2dw=(1-|\xi|^2)dz$, we have

$$\int_C \frac{ds_z}{|1-\xi^*z|^2}=\frac{1}{1-|\xi|^2}\int_{C'}ds_w$$

and

$$\int_C \frac{ds_z}{|z-\xi|^2}=\int_C\left|\frac{1-\xi^*z}{z-\xi}\right|^2\frac{ds_z}{|1-\xi^*z|^2}=\frac{1}{1-|\xi|^2}\int_{C'}\frac{ds_w}{|w|^2},$$

where C' is the curve into which C is transformed by (13). (12) is therefore equivalent to

$$(14)\qquad 0\leqq \Gamma(\xi,\xi)-K(\xi,\xi)\leqq\frac{1}{1-|\xi|^2}\left\{\int_{C'}\frac{ds_w}{|w|^2}-\left[\int_{C'}ds_w\right]^{-1}\right\}.$$

The expression on the right-hand side of (14) is—except for the factor $(1-|\xi|^2)^{-1}$—identical with the quantity estimated in Lemma II, with C replaced by C'. If $r=1+\epsilon q(\theta)$ is the polar equation of C_w, it thus follows from (14) and (11) that

$$(15)\qquad \begin{aligned} 0 &\leqq \Gamma(\xi,\xi)-K(\xi,\xi)\\ &\leqq\frac{\epsilon^2}{4\pi^2(1-|\xi|^2)}\left[\int_{C'}q_2(\theta)\,\frac{d\theta}{r}+\frac{1}{2}\int_{C'}q'^2(\theta)\left(\frac{1}{r}+\frac{1}{r^3}\right)d\theta\right].\end{aligned}$$

Because of $|\xi|(1+\epsilon M)<1$, the curve C' surrounds a finite region in the w-plane in the positive sense. Since C is in the annulus $1<z\leqq 1+\epsilon M$, C' will be in $1\leqq w\leqq 1+\epsilon M(1+|\xi|)[1-|\xi|(1+\epsilon M)]^{-1}$. Hence,

$$(16)\qquad 0\leqq q(\theta)\leqq\frac{M(1+|\xi|)}{1-|\xi|(1+\epsilon M)}.$$

We next have to obtain an estimate for $q'(\theta)$. Elementary geometric considerations show that $\epsilon q'(\theta)=r\tan\alpha$, where α is the angle between the radius-vector from the origin to the point $w=e^{i\theta}[1+\epsilon q(\theta)]$, and the normal to C' at the point w. Hence,

$$(17)\qquad \epsilon^2q'^2(\theta)=r^2\tan^2\alpha\leqq r^2\tan^2\alpha_0,$$

where $\alpha_0=\max|\alpha|$ for $0\leqq\theta<2\pi$. To obtain an upper bound for α, we have to express this angle in terms of geometric quantities related to the z-plane. In view of (13), a straight line through $w=0$ corresponds to a circle through $z=\xi$ which is orthogonal to $|z|=1$. Since the map (13) is conformal, α will thus be the angle between the normal to C—at a point z_0, say—and the orthogonal circle to $|z|=1$ which passes through ξ and z_0. Such a circle has an equation

(18) $$z = \frac{\xi + \kappa t}{1 + \xi^* \kappa t},$$

where κ is a constant such that $|\kappa| = 1$, and t is a real parameter. If z moves along this circle from ξ to z_0, t varies from 0 to a value $t_0 > 1$; evidently, the circle intersects $|z| = 1$ for $t = 1$. If we denote by β the angle between the normal to C at z_0 and the radius-vector from the origin to $z = z_0$, and by δ the angle between this radius-vector and the orthogonal circle C_0, then α will be the algebraic sum of β and δ.

To compute δ, we observe that

$$\delta = \left[\arg \frac{dz}{z}\right]_{z=z_0}, \qquad z \in C_0,$$

and therefore

$$\begin{aligned} -\delta &= \arg \frac{(1 + \xi^* \kappa t)(\xi + \kappa t)}{\kappa t} \\ &= \arg \{(\kappa^* + \xi^* t)(\xi + \kappa t)\} \\ &= \arg \{t(1 + |\xi|^2) + (1 + t^2) \operatorname{Re}\{\kappa^* \xi\} - i(t^2 - 1) \operatorname{Im}(\kappa^* \xi)\}. \end{aligned}$$

Hence,

(19) $$\tan \delta = \frac{(t^2 - 1) \operatorname{Im}\{\kappa^* \xi\}}{t(1 + |\xi|^2) + (1 + t^2) \operatorname{Re}\{\kappa^* \xi\}}.$$

Assuming that

(20) $$|\xi| t < 1,$$

and maximizing the right-hand side of (19) for all possible values of $\arg\{\kappa\}$, we find that

$$|\tan \delta| \leqq \frac{(t^2 - 1)|\xi|}{(t^2(1 - |\xi|^2)^2 - |\xi|^2(t^2 - 1)^2)^{1/2}}.$$

Setting, in accordance with (18),

$$t = \left|\frac{z - \xi}{1 - \xi^* z}\right|,$$

we obtain

$$|\tan \delta| \leqq \frac{|\xi|(|z|^2 - 1)}{(|z - \xi|^2 |1 - \xi^* z|^2 - |\xi|^2(|z|^2 - 1)^2)^{1/2}},$$

or, in view of

$$
\begin{aligned}
|(z-\xi)(1-\xi^*z)|^2 & \\
&\geqq (|z|-|\xi|)^2(1-|\xi|\,|z|)^2 \\
&= |z|^2(1-|\xi|)^4 - 2|\xi|(1+|\xi|^2)|z|(|z|-1)^2 \\
&\quad + |\xi|^2(|z|^2-1)^2,
\end{aligned}
$$

$$
|\tan\delta| \leqq \frac{|\xi|(|z|^2-1)}{(|z|^2(1-|\xi|)^4 - 2|\xi|\,|z|(1+|\xi|^2)(|z|-1)^2)^{1/2}}.
$$

Since, by (2), $r=|z|\leqq 1+\epsilon M$, this leads to

$$
(21) \qquad |\tan\delta| \leqq \frac{\epsilon M(1+r)}{(r^2(1-\eta)^4 - 2\epsilon^2M^2r\eta(1+\eta^2))^{1/2}},
$$

where $\eta\geqq|\xi|$. This estimate is valid if the expression under the radical is positive, which is certainly the case if (3) is satisfied. If (3) holds, the condition (20) will hold a fortiori.

The angle α—whose maximum appears in the inequality (17)—is the algebraic sum of δ and the angle $\beta=\arctan\{\epsilon p'(\theta)/r\}$, where $r=|z|=1+\epsilon p(\theta)$. In view of (21), it follows therefore that

$$
|\tan\alpha| \leqq \frac{\sigma+\tau/(A^2-B^2)^{1/2}}{1-\sigma\tau/(A^2-B^2)^{1/2}},
$$

where the abbreviations

$$
\sigma = \epsilon M'/r, \qquad \tau = \epsilon M(1+r),
$$
$$
A^2 = r^2(1-\eta)^4, \qquad B^2 = 2\epsilon^2M^2r\eta(1+\eta^2)
$$

have been employed. Squaring, and using the inequality $A(A^2-B^2)^{1/2} \geqq A^2-B^2$ $(A>B,\ A>0)$, we obtain after some simplifications

$$
\tan^2\alpha \leqq \frac{(\sigma A+\tau)^2-\sigma^2B^2}{(A-\sigma\tau)^2-B^2} \leqq \frac{(\sigma A+\tau)^2}{A^2-B^2-2A\sigma\tau}.
$$

Because of $0\leqq\eta<1$, $r\eta<1$, we have $(1-\eta)^2<1$ and $\eta(1+r)=\eta+\eta r<2$. Hence,

$$
\sigma A+\tau = \epsilon[M'(1-\eta)^2 + M\eta(1+r)] \leqq \epsilon(M'+2M)
$$

and

$$
\begin{aligned}
A^2 - B^2 - 2A\sigma\tau & \\
&= r[r(1-\eta)^4 - 2\epsilon^2\eta M\{M(1+\eta^2) + M'(1-\eta)^2(1+r^{-1})\}] \\
&\geqq r[(1-\eta)^4 - 4\epsilon^2M(M+M')].
\end{aligned}
$$

We may thus conclude from (17) that

$$\frac{\epsilon^2}{2}\left(\frac{1}{r}+\frac{1}{r^3}\right)q'^2(\theta) \leqq \frac{\epsilon^2 q'^2(\theta)}{r} \leqq r\tan^2\alpha$$
$$\leqq \frac{\epsilon^2(2M+M')^2}{(1-\eta)^4-4\epsilon^2M(M+M')}\cdot$$

Combining this with (15) and (16), we finally arrive at the inequality

$$\begin{aligned} &0 \leqq \Gamma(\xi,\xi) - K(\xi,\xi) \\ (22)\qquad &\leqq \frac{\epsilon^2}{2\pi(1-\eta^2)}\left[\frac{(2M+M')^2}{(1-\eta)^4-4\epsilon^2M(M+M')}+\frac{4M^2}{[1-\eta(1+\epsilon M)]^2}\right]. \end{aligned}$$

The right-hand side of (22) is identical with that of (5′), except for the factor ϵ^2. Since, in view of (4) and (7), $\epsilon^2R(z,\xi)=K(z,\xi)-\Gamma(z,\xi)$, the proof of Theorem I now follows from (5) and Lemma I.

We finally remark that Lemma II will also yield the estimate

$$(23)\qquad 0 \leqq \frac{1}{d}-\frac{2\pi}{L} \leqq \epsilon^2(M^2+M'^2),$$

for the outer conformal radius d of the domain D [3], where L denotes the length of the curve C. The outer conformal radius is defined by the expansion

$$w = F(z) = \frac{z}{d}+a_0+\frac{a_1}{z}+\cdots \qquad (d>0)$$

of the function mapping the complement of D onto $w>1$. Since D does not contain the point $z=0$, it follows that

$$\frac{1}{d} = -\frac{1}{2\pi i}\int_C F(z)\frac{dz}{z^2},$$

and thus

$$(24)\qquad \frac{1}{d} \leqq \frac{1}{2\pi}\int_C \frac{ds}{r^2}\cdot$$

On the other hand,

$$d = \frac{1}{2\pi i}\int_C \frac{dz}{F(z)},$$

and therefore [3]

$$d \leqq \frac{1}{2\pi}\int_C ds.$$

Combining this with (24), we obtain

$$0 \leqq \frac{1}{d} - \frac{2\pi}{L} \leqq \frac{1}{2\pi}\int_C \frac{ds}{r^2} - 2\pi \Big/ \int_C ds.$$

Since $r \geqq 1$, it follows from Lemma II that the right-hand side is bounded by $\epsilon^2(M^2+M'^2)$. This proves (23).

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