GEODESIC INSTABILITY

L. W. GREEN[1]

1. It has long been realized (see, for example, Morse [1][2]) that topological transitivity of the geodesic flow on a Riemannian manifold is implied by certain instability properties of the geodesics. If all curvatures are nonpositive, the required instability is immediate. However, when positive curvature is allowed, most of the results are limited to two-dimensions (Morse and Hedlund [1], Hopf [1]). Salenius has proved transitivity for a class of three-dimensional compact manifolds, using essentially the same methods as Morse and Hedlund. This method involves proving the instability property by means of the *dynamical* situation. In the following we generalize to arbitrary dimensions a previous result[3] which showed that instability follows from the *geometry*, in particular, from the assumption about the nonexistence of conjugate points.

Let M be a complete, simply-connected, n-dimensional Riemannian manifold of class C^r $(r \geqq 4)$ with the property that no geodesic of M contains two mutually conjugate points. Then any two points of M may be joined by one and only one geodesic segment, and the length of this segment will be called the distance between the points. Let the geodesic rays (images of a half-open interval) g and h be parametrized by means of arc-length: $g(t)$, $h(s)$, $0 \leqq s, t < \infty$. We say that g and h *diverge* if $\lim_{t\to\infty} g(t)h = \infty$ and $\lim_{s\to\infty} h(s)g = \infty$ (where $g(t)h$ denotes the distance from the point $g(t)$ to the set h). The geodesic rays emanating from a point P are said to be *uniformly divergent* if, for a sequence $s_i \to \infty$ and rays h, g_i with P as initial point, $\lim\inf_i h(s_i)g_i < \infty$ implies $\lim_i g_i = h$. Uniform divergence at P clearly implies the divergence of any two geodesics intersecting at P, but the converse is probably true only for two dimensions, where it is an immediate consequence of the Jordan curve theorem.

If γ is a bivector at the point P, $K(P, \gamma)$ will denote the Riemannian curvature in the direction γ. The principal theorem is

THEOREM 1. *If $K(P, \gamma) \geqq -A^2$, and no geodesic of M contains a*

Presented to the Society, December 28, 1953; received by the editors June 12, 1953 and, in revised form, July 20, 1954 and June 22, 1955.

[1] Work supported in part by ONR contract N6ori-105.

[2] Numbers in brackets refer to the bibliography.

[3] In our paper [1], referred to in the sequel as S.

pair of mutually conjugate points, then the geodesic rays from any point are uniformly divergent.

The proof of Theorem 1 will be given in §§2 and 3. We devote the remainder of this section to indicating some if its applications. M will continue to denote a simply-connected manifold with no conjugate points: G is a properly discontinuous group of isometries of M, and M/G is the manifold obtained by identifying points congruent under G.

THEOREM 2. *If G is abelian and $R=M/G$ is compact, then through every point of R there is a closed geodesic of the homotopy type associated with each generator of G.*

PROOF. Let P be a point of R, $\overline{P}$ a point of M covering P, and T a generator of G. The theorem asserts that there is a geodesic $\bar{g}$ through $\overline{P}$ such that $T(\bar{g})=\bar{g}$. Let $\bar{g}_1$ be the geodesic ray with initial point $\overline{P}$ which contains $T(\overline{P})$. From the compactness of R and the commutativity of G it is easily seen (see Busemann [1, 9.7]) that the distance between $\overline{Q}$ and $T(\overline{Q})$ is uniformly bounded for $\overline{Q}\in M$. Hence $\bar{g}_1$ and $T(\bar{g}_1)$ are geodesic rays which do not diverge. But these rays have the point $T(\overline{P})$ in common, so by Theorem 1, $T(\bar{g}_1)$ must be contained in $\bar{g}_1$. Taking $\bar{g}$ to be the extension of $\bar{g}_1$ to a complete (infinite in both directions) geodesic concludes the proof.

Busemann [2] has proved Theorem 2 in a more elementary fashion and for a wider class of spaces. However, because *uniform* divergence obtains in the spaces we consider, a stronger result is possible. Call a unit vector periodic if the (unique) geodesic to which it is tangent is periodic. Then we can prove

THEOREM 3. *Under the assumptions (and notation) of Theorem 2, the periodic vectors at any point of R are dense in the set of all unit vectors at that point.*

PROOF. Let u be a unit vector at the point P of R, and V a neighborhood of u in the unit sphere of tangent vectors. It is sufficient to consider the whole configuration in M, and to find a periodic vector $\bar{v}$ in the corresponding neighborhood V' of the sphere in the space tangent at a point $\overline{P}$ which covers P. (By abuse of language, we call a vector $\bar{v}$ on M periodic if the geodesic it determines is invariant under some motion of G.) Let $\bar{h}$ be the geodesic ray in M determined by $\bar{u}$, and designate by K the set of points (excluding $\overline{P}$) on all geodesic rays with initial point $\overline{P}$ and initial direction in V'. K is open and $\bar{h}(s)$, $s>0$, is contained in K. Because $\bar{u}$ is interior to V', the uniform divergence property of the rays with initial point $\overline{P}$ implies that the distance from $\bar{h}(s)$ to $M-K$ approaches infinity as s increases without bound. Consequently, for some value of s, say $\bar{s}$, that distance will exceed twice the diameter of R.

The copy of the fundamental domain of R which contains the point $\bar{h}(\bar{s})$ must then lie entirely in K. The proof of the theorem is completed by connecting $\overline{P}$ with the point congruent to it in that domain by a geodesic. This geodesic clearly has a periodic initial vector in V'.

We now specialize the manifold M to be the interior of the unit ball, U_n, in n-dimensional Euclidean space, endowed with the metric

$$ds^2 = 4f^2(x)dx_idx_i/(1 - x_ix_i)^2,$$

where $f(x) = f(x_1, x_2, \cdots, x_n)$ is of class C^4 and $0 < a \leqq f(x) \leqq b$ for all x in U_n and constant a, b. If G is a Fuchsian group, properly discontinuous in U_n, which leaves both the metric (1) and the hyperbolic metric ($f(x) \equiv 1$) invariant, we shall denote the manifold M/G by $M(G)$. The geodesic flow of $M(G)$ is defined to be the one-parameter group of homeomorphisms of the tangent sphere bundle of $M(G)$ which takes a unit vector e after "time" t into e_t, the unit tangent vector to the geodesic ray with initial point and direction e at a distance t from e (measured along the ray). The flow is said to be topologically transitive if there exists a vector the totality of whose images under the homeomorphisms is dense in the bundle. For the details of setting up the manifold $M(G)$ and the flow, we refer to Utz [**1**, §§1–6]. (Notice that under our blanket assumption of no conjugate points every geodesic of M is class A.) These same sections of Utz's paper also establish the preliminary results necessary for carrying out the argument of Theorem 13.1 of Morse and Hedlund [**1**], *provided the divergence of the geodesics in the covering space, M, is known.* In view of Theorem 1, we may therefore state

THEOREM 4. *If* (i) *G is of the first kind* (*ceases to be properly discontinuous at every point of the boundary of U_n*), (ii) *$K(P, \gamma) \geqq -A^2$ for A constant and every P, γ in $M(G)$, and* (iii) *no geodesic of $M(G)$ has a pair of mutually conjugate points, then the geodesic flow in $M(G)$ is topologically transitive.*

This result improves that of Salenius [**1**] in that $M(G)$ is no longer necessarily compact, the dimension is clearly arbitrary (although, as he indicates, his argument will carry over to higher dimensions), and the proof does not involve the Poincaré recurrence theorem.

2. **Jacobi equations.** The function $K(x)$ will be said to satisfy con-

ditions (C) if $K(x)$ is continuous, $K(x) \geqq -A^2$ for $-\infty < x < \infty$, and the equation

$$\text{(K)} \qquad y''(x) + K(x)y(x) = 0$$

has no nontrivial solution with more than one zero. (Equation (K) is the familiar Jacobi equation of geodesic variation for two-dimensional manifolds.)

LEMMA 1. *There exists an* $x_0 > 0$ *which depends on the conditions* (C) *but not on the specific function* $K(x)$ *satisfying these conditions, such that, if* $y(x)$ *is a solution of* (K) *with* $y(0) = 0$, $y'(0) > 0$, *then*

$$|y'(x)/y(x)| \leqq 2A, \qquad x \geqq x_0.$$

PROOF. Set $u(x) = y'(x)/y(x)$, for $x > 0$. Then $u(x)$ satisfies the Riccati equation

$$\text{(R)} \qquad u'(x) + u^2(x) + K(x) = 0.$$

Comparing with the equation

$$(\mathrm{R}_A) \qquad w'(x) + w^2(x) - A^2 = 0,$$

we find by a standard argument (see, e.g., *S*) that if $u(\bar{x}) = w(\bar{x}) > A$ for an $\bar{x} > 0$ in the domain of definition of $w(x)$, then $u'(\bar{x}) \leqq w'(\bar{x})$. Thus if $u(x_1) < w(x_1)$ for any x_1, then $u(x) \leqq w(x)$ for $x \geqq x_1$, by inspection of the manner of crossing of the integral curves. In particular, if $w(x, a)$ is the solution of (R_A) for which $\lim_{x \to a+} w(x, a) = +\infty$, $a > 0$, it is clear that $u(a+\epsilon) < w(a+\epsilon)$, for suitably small positive ϵ. Since a may be made as small as desired, $u(x) \leqq w(x, 0)$ for all $x > 0$. This solution of (R_A) depends only on A, and $\lim_{x \to \infty} w(x, 0) = A$. Hence we may conclude that $u(x) \leqq 2A$ for $x \geqq x_0$, where x_0 depends only on A. That $u(x) > -A$ for $x > 0$ follows from a similar argument and the fact that $u(x)$ is defined for all positive x (for details, see S). This completes the proof of the lemma.

We remark that if $y(x)$ is a solution of (K) which never vanishes the conclusion of the lemma holds for all x (in fact, for A instead of $2A$).

Consider the sequence of Jacobi equations

$$(\mathrm{K}_i) \qquad y_i''(x) + K_i(x)y_i(x) = 0,$$

where each K_i satisfies conditions (C). Then if $y_i(0) = 0$, $y_i'(0) > 0$, Lemma 1 says that $|y_i'(x)/y_i(x)| \leqq 2A$ for $x \geqq x_0$, where x_0 is independent of i. Assume now that $\lim_{i \to \infty} K_i(x) = K(x)$ uniformly on every compact interval, and let $K(x)$ satisfy conditions (C). Then if

$y_i(x)$ are solutions of (K_i) with the same boundary conditions, $y_i(x) \to y(x)$ uniformly on bounded intervals, where $y(x)$ is the solution of (K) with the prescribed boundary conditions. Here by "boundary conditions" we mean conditions imposed on the $y_i(x)$ and (or) their derivatives at finite points. To extend this convergence to solutions satisfying a special type of infinite boundary value is the purpose of the following paragraphs.

Let $y_i(x)$, $w_i(x, a)$ be solutions of (K_i) for which $y_i(0) = 0 = w_i(a, a)$, $y_i'(0) = 1 = w_i(0, a)$. Then for $x > 0$

$$w_i(x, a) = y_i(x) \int_x^a y_i^{-2}(s) ds,$$

and, as is well known (see, for example, E. Hopf [1]), $\lim_{a\to\infty} w_i(x, a) = u_i(x)$ exists, is a solution of (K_i) which never vanishes, and may be represented for $x > 0$ by the equation

$$u_i(x) = y_i(x) \int_x^\infty y_i^{-2}(s) ds. \tag{1}$$

Since $w_i'(0, 1) < u_i'(0) < w_i'(0, -1)$ for all i, there exists a subsequence such that $\lim_{k\to\infty} u_{i_k}'(0) = u'$ exists. It is then easy to see that $\lim_{k\to\infty} u_{i_k}(x) = u(x)$ exists and is a solution of (K) satisfying the boundary conditions $u(0) = 1$, $u'(0) = u'$. Moreover, $u(x) > 0$ for $x \geqq 0$.

LEMMA 2. *If y_i is the solution of (K_i) with $y_i(0) = 0$, $y_i'(0) = 1$, then there exists no constant R and sequence x_i such that* $\lim_i x_i = \infty$ *but* $y_i(x_i) \leqq R$ *for all i.*

PROOF. Choose $x_0 > 0$ and set $\inf_i \{y_i(x_0)\} = c$. Since $\lim_i y_i(x) = y(x)$, $c > 0$. Moreover, since $\lim_{k\to\infty} u_{i_k}(x) = u(x) > 0$, it is possible to choose $b > 0$ such that the (never-vanishing) solutions of (K_{i_k}) defined by $h_{i_k}(x) = bu_{i_k}(x)$ satisfy $h_{i_k}(x_0) \leqq c \leqq y_{i_k}(x_0)$. The Sturmian separation theorem then implies that $h_{i_k}(x) < y_{i_k}(x)$ for $x > x_0$. The numbers $a_{i_k} = h_{i_k}(x_{i_k})/y_{i_k}(x_{i_k})$ are bounded so that, using the hypothesis, we may find a subsequence of $\{i_k\}$ (to which we confine attention in the following, and therefore designate with the single subscript m) such that $\lim_m a_m = a$ and $\lim_m y_m(x_m) = R'$. Form the solution $z_m(x)$ of (K_m) by setting

$$z_m(x) = h_m(x) - ay_m(x).$$

Then $\lim_m z_m(x_m) = 0$. But calculation of the Wronskian shows that

$$y_m'(x)z_m(x) - z_m'(x)y_m(x) = z_m(0) = b > 0. \tag{2}$$

Lemma 1 implies that $y_m'(x)/y_m(x)$ and $z_m'(x)/z_m(x)$ are uniformly bounded for sufficiently large x. It follows that $y'_m(x_m)$ is a bounded sequence. But dividing (2) by $z_m(x_m)$ we get

$$y'_m(x_m) - y_m(x_m)z'_m(x_m)/z_m(x_m) = b/z_m(x_m).$$

Since the left side is bounded while the right is not, we obtain the contradiction which proves the lemma.

It is clear that the results of this section hold if, instead of requiring that no solution vanish twice, we merely assume that there is an $\epsilon > 0$ (independent of i) such that any solution with a zero in $(-\epsilon, \epsilon)$ has no other zero.

Morse in [1] paraphrases his assumption of *uniform instability* as the hypothesis that the first conjugate point of each finite point lies "beyond" the point at infinity. Lemma 2 shows that this is implied (and uniformly so) merely by assumptions (C). Setting all the K_i equal, we have the result of S as a

COROLLARY. *If* $y(x)$ *is a solution of* (K) *with* $y(0) = 0$, $y'(0) > 0$, *then* $\lim_{x\to\infty} y(x) = \infty$.

3. **Proof of Theorem** 1. Suppose the theorem false, that is, suppose there is a point P in the manifold M described in §1 which is the initial point of the geodesic rays g_k, g, and h, and that $g_k \to g \neq h$, yet there exists a sequence $s_k \to \infty$ such that

$$\lim_k \inf g_k h(s_k) < \infty.$$

It is clear that $g(1) \neq h(-1)$ (where negative values of the arc-length parametrize the oppositely directed ray), so that we may assume that a subsequence of the g_k's has already been chosen in such a manner that neither $h(-1)$ nor $h(1)$ is a limit point of $\{g_k(1)\}$. If S denotes the unit (geodesic) sphere with center P, it is then possible to set up polar coordinates (r, y) in M, where $y = (y^1, \cdots, y^{n-1})$ are coordinates on S valid in $S - h(-1)$. The line element of the space then takes the form (repeated Greek letters will always be summed from 1 to $n-1$)

$$ds^2 = dr^2 + a_{\alpha\beta}(r, y)dy^\alpha dy^\beta.$$

In these coordinates the equations of the geodesic rays may be taken as $h(r) = (r, y_0)$, $g_k(r) = (r, y_k)$. Let $l_k(t) = (r_k(t), y_k(t))$ be a geodesic segment whose length affords a minimum to the distance from $(r_k(0), y_k(0) = (s_k, y_0) = h(s_k)$ to g_k, parametrized so that $0 \leqq t \leqq 1$. Set $\bar{l}_k(t) = (1, y_k(t))$; that is, $\bar{l}_k$ is the projection of l_k on S. If x_k is the arc length along $\bar{l}_k$ measured from $h(1)$, then $t_k = t_k(x_k)$ is a legitimate change of parameter of l_k and $\bar{l}_k$. Setting $z_k(x_k) = y_k(t_k(x_k))$, $R_k(x_k)$

$=r_k(t_k(x_k))$, we may therefore designate these lines by the equations $l_k(x_k)=(R_k(x_k), z_k(x_k))$ and $\bar{l}_k(x_k)=(1, z_k(x_k))$. Here the parameter x_k has the domain $[0, \bar{L}_k]$, where $\bar{L}_k$ is the length of $\bar{l}_k$. Since $h(1)$ is not a limit point of $g_k(1)$, $h(1)$ has a neighborhood of geodesic radius L_0 which contains no point $g_k(1)$ for sufficiently large k. Then $\bar{L}_k \geqq L_0 > 0$ for large k, so we may drop the subscript from x_k if we confine it to the interval $[0, L_0]$. Because of the choice of parameter,

$$a_{\alpha\beta}(1, z_k(x))\dot{z}_k^{\alpha}\dot{z}_k^{\beta} = 1,$$

where the dot denotes differentiation with respect to x and only repeated Greek letters are summed. Since the closed ball Σ of radius L_0 about $h(1)$ is compact and does not contain $h(-1)$, there exists a positive constant B such that

$$a_{\alpha\beta}(1, y)u^{\alpha}u^{\beta} \leqq B^2 u^{\alpha}u^{\alpha}$$

for all y in $\Sigma \cap S$ and all u. In particular, setting

$$c_k^2(x) = \dot{z}_k^{\alpha}(x)\dot{z}_k^{\alpha}(x),$$

$$1 \leqq B^2 \dot{z}_k^{\alpha}\dot{z}_k^{\alpha} = B^2 c_k^2(x),$$

so that $c_k^2(x) \geqq B^{-2} > 0$ for all x in $[0, L_0]$, and B does not depend on k.

If $L_k = gh(s_k)$, that is, L_k is the length of l_k, we have that

$$L_k = \int_0^{\bar{L}_k} \{\dot{R}_k^2 + a_{\alpha\beta}(R_k, z_k)\dot{z}_k^{\alpha}\dot{z}_k^{\beta}\}^{1/2} dx \geqq \int_0^{L_0} \{a_{\alpha\beta}(R_k, z_k)\dot{z}_k^{\alpha}\dot{z}_k^{\beta}\}^{1/2} dx.$$

The denial of the theorem is the statement that $\liminf_k L_k$ is finite. An application of Fatou's lemma then implies that

$$\liminf_k \{a_{\alpha\beta}(R_k(x), z_k(x))\dot{z}_k^{\alpha}\dot{z}_k^{\beta}\}^{1/2}$$

is integrable over the interval $[0, L_0]$. Hence there is an $\bar{x} \in (0, L_0)$ and a subsequence, which we shall denote with subscript i, such that

$$\{a_{\alpha\beta}(R_i(\bar{x}), z_i(\bar{x}))\dot{z}_i^{\alpha}(\bar{x})\dot{z}_i^{\beta}(\bar{x})\}^{1/2} \leqq D \tag{3}$$

for all i and a suitable constant D.

Our efforts will now be directed to proving that (3) leads to a contradiction. Let

$$Z_i^2(r) = a_{\alpha\beta}(r, z_i(\bar{x}))\dot{z}_i^{\alpha}(\bar{x})\dot{z}_i^{\beta}(\bar{x}).$$

Then $Z_i(0)=0$, and, denoting differentiation with respect to r by primes,

$$Z_i'(0) = \lim_{r\to 0} r^{-1}\left\{a_{\alpha\beta}z_i^\alpha z_i^\beta\right\}^{1/2} = \left\{\dot z_i^\alpha \dot z_i^\beta\right\}^{1/2} = c_i(\bar x) \geqq B^{-1}.$$

In these polar coordinates the Christoffel symbols and the components of the Riemann tensor in which we are interested are given by

$$\Gamma_{\alpha\ \beta}^{\ n} = \Gamma_{\alpha n\beta} = -\Gamma_{n\alpha\beta} = -\frac{1}{2}\frac{\partial a}{\partial r}\alpha\beta,$$

and

$$R_{n\alpha n\beta} = \frac{\partial}{\partial r}\Gamma_{\alpha n\beta} - \Gamma_{\alpha n\gamma}\Gamma_{\beta\ n}^{\ \gamma}.$$

Differentiating $Z_i(r)$ twice, we find that

$$Z_i'(r) = \frac{1}{2}\left\{a_{\alpha\beta}\dot z_i^\alpha \dot z_i^\beta\right\}^{-1/2}\frac{\partial a_{\gamma\delta}}{\partial r}\dot z_i^\gamma \dot z_i^\delta = -Z_i^{-1}\Gamma_{\alpha n\beta}\dot z_i^\alpha \dot z_i^\beta,$$

and

$$Z_i''(r) = -Z_i^{-1}\frac{\partial}{\partial r}\Gamma_{\alpha n\beta}\dot z_i^\alpha \dot z_i^\beta - Z_i^{-3}\left\{\Gamma_{\sigma n\delta}\dot z_i^\sigma \dot z_i^\delta\right\}^2$$

$$= -Z_i^{-1}R_{n\alpha n\beta}\dot z_i^\alpha \dot z_i^\beta - Z_i^{-1}\Gamma_{\sigma n\gamma}\Gamma_{\beta\ n}^{\ \gamma}\dot z_i^\sigma \dot z_i^\beta - Z_i^{-3}\left\{\Gamma_{\sigma n\delta}\dot z_i^\sigma \dot z_i^\delta\right\}^2.$$

More concisely,

$$(\mathrm{J}_i) \qquad Z_i''(r) + [\overline{K}_i(r) - T_i(r)]Z_i(r) = 0,$$

where $\overline{K}_i(r) = Z_i^{-2}R_{n\alpha n\beta}\dot z_i^\alpha \dot z_i^\beta$, the curvature of the orthogonal bivector formed by the tangent to the geodesic $(r, z_i(\bar x))$ and the vector $(\dot z_i(\bar x))$. If we set $v_\alpha^i = \Gamma_{\alpha n\beta}\dot z_i^\beta$, $T_i(r)$ may be written in a simple form:

$$T_i = Z_i^{-4}\left\{Z_i^2\Gamma_{\alpha n\sigma}a^{\sigma\delta}\Gamma_{\beta n\delta}\dot z_i^\alpha \dot z_i^\beta - \left[\Gamma_{\sigma n\delta}\dot z_i^\sigma \dot z_i^\delta\right]^2\right\}$$

$$= Z_i^{-4}\left\{Z_i^2V_i^2 - (v_\alpha^i \dot z_i^\alpha)^2\right\} = V_i^2Z_i^{-2}\sin^2\theta_i,$$

where V_i is the length of (v^i) and θ_i is the angle this vector makes with $(\dot z_i)$.

Now choose a subsequence of the integers, which will be denoted by the subscript j, such that $\lim_{j\to\infty} z_j(\bar x)$ and $\lim_{j\to\infty} \dot z_j(\bar x)$ both exist. (This is possible because $\Sigma\cap S$ is compact and the $\dot z$'s are unit vectors.) Call these limits $\bar z$ and $\bar w$, respectively. Setting $K_j = \overline{K}_j - T_j$, we have that $\lim_{j\to\infty} K_j(r) = K(r)$ uniformly for $0 < \epsilon \leqq r \leqq 1/\epsilon$, since all the quantities involved are uniformly continuous in this range, and depend continuously on (z) and $(\dot{z})$. (ϵ is any fixed positive constant.) Moreover, $K(r)$ may be defined for all r, and seen to be bounded in a neighborhood of the origin, by considering the equation

$$\text{(J)} \qquad Z''(r) + K(r)Z(r) = 0,$$

where $Z(r) = \lim_{j\to\infty} \{a_{\alpha\beta}(r, z_j)\dot{z}_j^\alpha \dot{z}_j^\beta\}^{1/2} = \{a_{\alpha\beta}(r, \bar{z})\bar{w}^\alpha \bar{w}^\beta\}^{1/2}$. Similarly, $K_j(r)$ (although not necessarily $\overline{K}_j$ and T_j separately) are uniformly bounded in any fixed neighborhood of the origin, by virtue of the "in-the-large" definition of $Z_j(r)$ and its behavior at $r=0$.

The nonconjugacy hypothesis assures us that $Z_j(r)$, $Z(r)$ vanish only for $r=0$, so none of the equations (J_j), (J) has a solution with more than one zero.

Let $y_j(r) = Z_j(r)/c_j(\bar{x})$. Then $y_j'(0) = 1$, and

$$y_j(R_j(\bar{x})) = Z_j(R_j(\bar{x}))/c_j(\bar{x}) \leqq BZ_j(R_j(\bar{x})) \leqq BD,$$

by (3) and the definition of Z_j. If we could apply Lemma 2 to this situation, we would have the desired contradiction. The only hypothesis which is as yet unverified is the boundedness condition, and, because of the above remarks about $K_j(r)$ on compact intervals and our blanket assumption on the curvatures, it is sufficient to show that the quantities $T_j(r)$ are uniformly bounded for r sufficiently large.

First we notice that $\Gamma_{\alpha n\beta}(r)$ are the coefficients of the second fundamental form of the geodesic hypersphere $S(P, r)$ (center P and radius r) at the point where the geodesic we consider pierces it. (We omit the index j—what follows will be true uniformly in j.) This fact results from a direct computation, which is carried out in, for example, Cartan [1, p. 228]. Consider $\Gamma_{\alpha n\beta}$ as the matrix of a symmetric linear transformation in the $(n-1)$-dimensional vector space tangent to $S(P, r)$. We shall prove that these transformations, and, hence, $T(r)$, are bounded uniformly in r relative to the metric induced in these vector spaces by $(a_{\alpha\beta})$. Since the matrices are symmetric, a standard argument (for example, on the eigenvalues) shows that it is sufficient to prove that the associated quadratic forms are bounded.

This has already been done by Rauch [1, p. 43, Lemma 2], but because of the geometric significance it seems not altogether useless to give a different, more geometric proof.

LEMMA 3 (RAUCH). *There exists an r_0 such that, for $r \geqq r_0$, the second fundamental forms $\Gamma_{\alpha n\beta}(r)u^\alpha u^\beta$ are uniformly bounded.*

PROOF. Let (u) be any unit vector tangent to $S(P, r)$, and consider the two-dimensional (local) Riemannian submanifold V_2 obtained by translating (u) parallel to itself along the geodesic g through P and the point of tangency. The intersection of $S(P, r)$ and V_2 is a curve Γ whose geodesic curvature k_Γ (relative to V_2) at the point Q where it crosses g is precisely $\Gamma_{\alpha n\beta}u^\alpha u^\beta$. The arc of the geodesic (relative to the metric induced in V_2) circle C with center P and tangent to Γ at Q lies entirely on one side of Γ in the neighborhood of Q, since the distance from P to a point of Γ measured in V_2 is at least as great as r. Hence the curvature k_C of C at Q satisfies the inequality $k_C \geqq k_\Gamma$. Now if $ds^2 = dr^2 + G^2(r, v)dv^2$ is the line element of V_2 in geodesic coordinates with g as base, we have that

$$k_C = G^{-1}(r, 0) \frac{\partial}{\partial r} G(r, 0).$$

Then

$$k_C' + k_C^2 + V = 0, \tag{4}$$

where $V(r)$ is the intrinsic Gaussian curvature $(= -G^{-1}\partial^2/\partial r^2 G)$ of V_2. Notice that $k_C(r)$ is a solution of (4) which exists for all $r > 0$; moreover there are no points on g conjugate to P in V_2, since, a fortiori, g is a minimizing curve in the imbedding space. Finally, $V(r) \geqq -A^2$, for when a V_2 is formed by parallel displacement of a vector along a geodesic, the Gaussian curvature along this curve equals the Riemannian curvature of the tangent bivectors. (This is the so-called "Lemma of Synge"; cf. Preissman [1].) Therefore we conclude, by Lemma 1, that there exist numbers r_0 and T_0, which depend only on A, such that, if $r \geqq r_0$, $|k_C(r)| \leqq T_0$. Hence $k_\Gamma \leqq T_0$ for $r \geqq r_0$. (The more specific comparison with the space of constant curvature which is actually Rauch's Lemma may be obtained by a more careful examination of Lemma 1.)

Because we have made no assumption about the convexity of the spheres, we cannot conclude that k_Γ is non-negative. Nor will the above method give us a lower bound, even if we assume that the Riemannian curvatures are bounded above by some positive quantity, since the question of conjugate points in the comparison space would then arise. Fortunately, no additional assumption is necessary. Let P' be the point on g such that Q bisects the arc from P to P', and let C' be the geodesic circle in V_2 with center P' which passes through Q. We assert that, in the neighborhood of Q, Γ is between C and C'. For, shifting attention back to the entire manifold, the spheres $S(P, r)$ and $S(P', r)$ have only the point Q in common (because of the uniqueness of geodesics) and C' is clearly inside $S(P', r)$.

Hence $-k_\Gamma \leqq k_{C'} \leqq T_0$ for $r \geqq r_0$, by the same reasoning used for k_C. This completes the proof of the lemma, and the theorem.

Theorem 1 is slightly less general than its two-dimensional analogue (proved in 3). For in two dimensions it is necessary to assume only that P is interior to the set of poles of M (a pole is a point conjugate to no other point), while in the above proof we have made use of the fact that P', at an arbitrary distance from P, also was a pole. However, an examination of the proof shows that only the fact that P' had no conjugate point on the geodesic segment connecting it with P was used, and this follows from the weaker assumption. Hence we may state

THEOREM 1′. *If $K(Q, \gamma) \geqq -A^2$ for some constant A and all Q, γ of M, and P is interior to the set of poles of M, then the geodesic rays from P are uniformly divergent.*

BIBLIOGRAPHY

H. BUSEMANN

1. *Spaces with nonpositive curvature*, Acta Math. vol. 80 (1948) pp. 259–310.
2. *Motions with maximal displacements*, Comment. Math. Helv. vol. 28 (1954) pp. 1–8.

E. CARTAN

1. *Leçons sur la géométrie des espaces de Riemann*, Paris, 1946.

L. W. GREEN

1. *Surfaces without conjugate points*, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 529–546.

E. HOPF

1. *Statistik der Lösungen geodätischer Problem vom unstabilen Typus* (II), Math. Ann. vol. 117 (1940) pp. 590–608.

M. MORSE

1. *Instability and transitivity*, J. Math. Pures Appl. (9) vol. 14 (1935) pp. 49–71.

M. MORSE and G. A. HEDLUND

1. *Manifolds without conjugate points*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 362–386.

A. PREISSMAN

1. *Quelques propriétés globales des espaces de Riemann*, Comment. Math. Helv. vol. 15 (1942) pp. 175–216.

H. E. RAUCH

1. *A contribution to differential geometry in the large*, Ann. of Math. vol. 54 (1951) pp. 38–55.

T. SALENIUS

1. *Ueber geodätische Linien in gewissen dreidimensionalen Mannigfaltigkeiten*, Annales Academiae Scientiarum Fennicae (A) vol. 47 (1948) pp. 1–52.

W. R. UTZ

1. *Almost periodic geodesics on manifolds of hyperbolic type*, Duke Math. J. vol. 18 (1951) pp. 147–164.

PRINCETON UNIVERSITY AND
UNIVERSITY OF MINNESOTA